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Chapter 12
Analysis of a Network's Emerging Behaviour via Its Structure Involving Its Strongly Connected Components

Abstract In this chapter, it is addressed how network structure can be related to network behaviour. If such a relation is studied, that usually concerns only strongly connected networks and only linear functions describing the aggregation of multiple impacts. In this chapter both conditions are generalised. General theorems are presented that relate emerging behaviour of a network to the network's structure characteristics. The network structure characteristics on the one hand concern network connectivity in terms of the network's strongly connected components and their mutual connections; this generalises the condition of being strongly connected (as addressed in Chap. 11) to a very general condition. On the other hand, the network structure characteristics considered concern aggregation by generalising from linear combination functions to any combination functions that are normalised, monotonic and scalar-free, so that many nonlinear functions are also covered (which also was done in Chap. 11). Thus the contributed theorems generalise existing theorems on the relation between network structure and network behaviour that only address specific cases (such as acyclic networks, fully and strongly connected networks, and theorems addressing only linear functions).

12.1 Introduction

In many cases, the relation between network structure and its emerging behaviour is only studied by performing simulation experiments. In this chapter, it is shown how within a certain context it is also possible to analyse mathematically how certain behaviours relate to certain properties of the network structure. For the network's structure, two types of characteristics are considered: (1) characteristics of the network's connectivity, and (2) characteristics of the aggregation in the network. In Chap. 11, the above question was only addressed for quite specific connectivity characteristics, in particular, for the case of an acyclic network, and the case of a strongly connected network. For these connectivity characteristics, the current chapter uses the general setting based on the strongly connected components of the network to develop a mathematical analysis for the general case. Tools were adopted from the area of Graph Theory, in particular, the manner to identify the

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connectivity structure within a graph by decomposition of the graph according to its (maximal) strongly connected components and the resulting (acyclic) condensation graph (Harary et al. 1965), Chap. 3, and in addition the notion of stratification of an acyclic directed graph; e.g., Chen (2009).

Besides the connectivity characteristics of the network structure, the theorems presented here also take into account the aggregation characteristics of the network structure, by identifying relevant properties of the combination functions by which the impacts from multiple incoming connections are aggregated. It applies not to just one most simple type of (for example, linear) functions, but to a wider class of functions: those combination functions that are characterised as being monotonic, scalar-free and normalised. These properties of combination functions already turned out important in Chap. 11 for acyclic and strongly connected networks, and will also turn out to be important for the general case concerning the connectivity characteristics. This class of functions includes not only the often used linear functions, but also nonlinear functions such as *n*th order Euclidean combination functions and normalised scaled geometric mean functions.

The theorems explain which are the relevant characteristics that make that these combination functions contribute to certain behaviour when $t \rightarrow \infty$. It will be shown how using the above mentioned tools from Graph Theory, together with the aggregation characteristics of combination functions mentioned, enable to address the general case and obtain theorems about it. These theorems apply to arbitrary types of networks, but among the foci of application, in particular, are the types of example network models of which several are described in Treur (2016b):

(1) Mental Networks describing the dynamics of mental processes as the (usually cyclic) interaction of the mental states involved, and behaviour resulting from this,
(2) Social Networks describing social contagion processes for opinions, beliefs, emotions, for example,
(3) Integrative Networks that integrate (1) and (2).

Note that especially Mental Networks are often not strongly connected, although some parts may be. Typically they use sensory input that in general may not be affected by the behaviour, and because of that such input is not on any cycle of the network. Therefore they cannot be treated like strongly connected networks, but the theory developed here based on a decomposition by strongly connected components does apply (for applying the analysis to an example of such a Mental Network, see Sect. 12.7.2 below). Social Networks may often be strongly connected, but also in that case external nodes may be involved that affect them, which makes the whole network not strongly connected. Therefore for applicability on such types of networks the generalisation from strongly connected networks to general types of networks is important.

The foci of applicability on the three types of networks (1)–(3) mentioned above also makes that only addressing linear functions would be too limited. Especially for Mental Networks, often nonlinear functions are used. Therefore the challenge is also to stretch the type of analysis to at least certain types of nonlinear functions.

To apply the theorems introduced in this chapter to any given network, first the decomposition of the network into its strongly connected components is determined. Multiple efficient algorithms are available to determine these strongly connected components; e.g., see Bloem et al. (2006), Fleischer et al. (2000), Gentilini et al. (2003), Li et al. (2014), Tarjan (1972), Wijs et al. (2016), Lacki (2013). The connections between these components are identified, as represented in an acyclic condensation graph, and stratification of this graph is introduced. Based on this acyclic and stratified structure added to the original network, the theorems will show whether and which states within the network will end up in a common equilibrium value, and more in general determine bounds for the equilibrium values of the states.

The research presented here has been initiated from the angle of mathematical analysis and verification of network models in comparison to simulations for these models. For more background on this angle, see, for example, Treur (2016a) or Treur (2016b), Chap. 12. Like verification in Software Engineering is very useful for the quality of developed software e.g., Drechsler (2004), Fisher (2007), verification in network modeling is a useful means to get implementations of network models in accordance with the specifications of the models, and eliminate implementation errors. If a simulation of an implemented network model contradicts one or more of the results presented in the current chapter for the specification of the network model, then this pinpoints that something is wrong: a discrepancy between specification and implementation of the network model that needs to be addressed. Afterwards, it turned out that the contributions presented here also have some relations to research conducted from a different angle, namely on control of networks; e.g., Liu et al. (2011, 2012), Moschoyiannis et al. (2016), Haghighi and Namazi (2015), Karlsen and Moschoyiannis (2018). These relations will be discussed in the Discussion section.

In Sect. 12.2 the basic definition of network used is summarised. Section 12.3 discusses emerging behaviour, illustrated for an example network. Section 12.4 presents the definitions of the Graph Theory tools for the considered network connectivity characteristics; in Sect. 12.5 the identified aggregation characteristics in terms of combination functions are defined. In Sect. 12.6 the main theorems are formulated and it is pointed out how they were proven, thereby referring to Chap. 15, Sect. 15.7 for more complete proofs. In Sect. 12.7 more in-depth analysis is added, and in particular, applicability is illustrated for a type of network which is not a Social Network: a Mental Network describing sharing behaviour based on emotional charge. Section 12.8 is a final discussion.

12.2 Temporal-Causal Networks

This section describes the definition of the concept of network model used: temporal-causal network model. This is a notion of network that covers all types of discrete and smooth continuous dynamical systems, as has been shown in Treur (2017), building further, among others, on Ashby (1960) and Port and van Gelder (1995).

A temporal-causal network model is based on three notions defining the network structure characteristics: connection weight (Connectivity), combination function (Aggregation), and speed factor (Timing); see Table 12.1, upper part. Here the word temporal in temporal-causal refers to the causality. A library with a number (currently 35) of standard combination functions is available as options to choose from; but also own-defined functions can be used.

In Table 12.1, lower part it is shown how a conceptual representation of network structure defines a numerical representation of network dynamics; see also (Treur 2016b), Chap. 2, or (Treur 2019). Here $X_1, \ldots, X_k$ with $k \geq 1$ are the states from which state Y gets its incoming connections. This defines the detailed dynamic semantics of a temporal-causal network. Note that in the current chapter all connection weights are assumed nonnegative.

The difference equations in the last row in Table 12.1 can be used for simulation and mathematical analysis. They can also be written in differential equation format:

$$\mathbf{d}Y(t)/\mathbf{d}t = \boldsymbol{\eta}_Y[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t),\ldots,\boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)] \tag{12.1}$$

Table 12.1 Conceptual and numerical representations of a temporal-causal network

Concepts	Notation	Explanation
States and connections	X, Y, $X \rightarrow Y$	Describes the nodes and links of a network structure (e.g., in graphical or matrix format)
Connection weight	$\boldsymbol{\omega}_{X,Y}$	*Connection weight* $\boldsymbol{\omega}_{X,Y} \in [-1, 1]$ represents the strength of the impact of state X on state Y through connection $X \rightarrow Y$
Aggregating multiple impacts	$\mathbf{c}_Y(..)$	For each state Y a *combination function* $\mathbf{c}_Y(..)$ is chosen to combine the causal impacts of other states on state Y
Concepts	Numerical representation	Explanation
State values over time t	$Y(t)$	At each time point t each state Y has a real number value, usually in [0, 1]
Single causal impact	$\mathbf{impact}_{X,Y}(t) = \boldsymbol{\omega}_{X,Y}\, X(t)$	At t state X with connection to state Y has an impact on Y, using weight $\boldsymbol{\omega}_{X,Y}$
Aggregating multiple impacts	$\mathbf{aggimpact}_Y(t) = \mathbf{c}_Y(\mathbf{impact}_{X_1,Y}(t), \ldots, \mathbf{impact}_{X_k,Y}(t)) = \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t))$	The aggregated impact of $k \geq 1$ states $X_1, \ldots, X_k$ on Y at t, is determined using combination function $\mathbf{c}_Y(..)$
Timing of the causal effect	$Y(t + \Delta t) = Y(t) + \boldsymbol{\eta}_Y [\mathbf{aggimpact}_Y(t) - Y(t)]$ $\Delta t = Y(t) + \boldsymbol{\eta}_Y [\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)]\, \Delta t$	The impact on Y is exerted over time gradually, using speed factor $\boldsymbol{\eta}_Y$

Note that combination functions usually are functions on the 0–1 interval within the real numbers: $[0, 1]^k \rightarrow [0, 1]$. Moreover, note that the condition $k \geq 1$ in Table 12.1 makes that by definition the above general format only applies to states Y with at least one incoming connection. However, in a network there also may be states Y without any incoming connection; for example, such states can serve as external input. Their dynamics can be specified in an independent manner by any mathematical function f: $[0, \infty) \rightarrow [0, 1]$ over time t:

$$Y(t) = f(t) \quad \text{for all} \quad t \tag{12.2}$$

Special cases of this are states Y with constant values over time, where for some constant c $\in [0, 1]$ it holds $f(t)$ = c for all t. For such constant states, still the general format can be used as well, as long as the speed factor $\boldsymbol{\eta}_Y$ is set at 0 and the combination function is well-defined for zero arguments: then the general format reduces to $Y(t + \Delta t) = Y(t)$, and therefore the initial value is kept over time. But there are also other possible types of external input, for example, a repeated alternation of values 0 and 1 for some time intervals to model episodes in which a stimulus occurs and episodes in which it does not.

Examples of often used combination functions (see also Treur 2016b, Chap. 2, Table 2.10) are the following:

- The *identity* function **id(.)** for states with only one impact

$$\mathbf{id}(V) = V$$

- the *scaled sum* function **ssum**$_\lambda$**(..)** with scaling factor $\boldsymbol{\lambda}$

$$\mathbf{ssum}_\lambda(V_1, \ldots, V_k) = \frac{V_1 + \cdots + V_k}{\boldsymbol{\lambda}}$$

- the scaled minimum function **smin**$_\lambda$**(..)** with scaling factor $\boldsymbol{\lambda}$

$$\mathbf{smin}_\lambda(V_1, \ldots, V_k) = \frac{\min(V_1, \ldots, V_k)}{\boldsymbol{\lambda}}$$

- the scaled maximum function **smax(..)** with scaling factor $\boldsymbol{\lambda}$

$$\mathbf{smax}_\lambda(V_1, \ldots, V_k) = \frac{\max(V_1, \ldots, V_k)}{\boldsymbol{\lambda}}$$

- the *simple logistic sum* combination function **slogistic**$_{\sigma,\tau}$**(..)** with steepness $\boldsymbol{\sigma}$ and threshold $\boldsymbol{\tau}$, defined by

$$\mathbf{slogistic}_{\sigma,\tau}(V_1, \ldots, V_k) = \frac{1}{1 + \mathrm{e}^{-\boldsymbol{\sigma}(V_1 + \cdots + V_k - \boldsymbol{\tau})}}$$

- the *advanced logistic sum* combination function $\mathbf{alogistic}_{\sigma,\tau}(..)$ with steepness σ and threshold τ, defined by

$$\mathbf{alogistic}_{\sigma,\tau}(V_1,\ldots,V_k) = \left[\frac{1}{1+\mathrm{e}^{-\sigma(V_1+\cdots+V_k-\tau)}} - \frac{1}{1+\mathrm{e}^{\sigma\tau})}\right](1+\mathrm{e}^{-\sigma\tau})$$

- the *Euclidean combination function* of nth order with scaling factor λ (generalising the scaled sum $\mathbf{ssum}_{\lambda}(..)$ for $n = 1$) defined by

$$\mathbf{eucl}_{n,l}(V_1,\ldots,V_k) = \sqrt[n]{\frac{V_1^n + \cdots + V_k^n}{\lambda}}$$

  Here n can be any positive integer, or even any positive real number.
- the *scaled geometric mean combination function* with scaling factor λ

$$\mathbf{sgeomean}_{\lambda}(V_1,\ldots,V_k) = \sqrt[k]{\frac{V_1 * \ldots * V_k}{\lambda}}$$

For example, scaled minimum and maximum functions are often used in fuzzy logic inspired modelling and modeling uncertainty in AI, and the logistic sum functions are often used in neural network inspired modeling. The scaled sum functions, which are a special (linear) case of Euclidean functions, are often used in modeling of social networks. Geometric mean combination functions relate to product-based combination rules often used for probability-based approaches.

Recall from Chap. 11 the picture shown in Fig. 12.1. It also applies here. The basic relation between structure and dynamics is indicated by the horizontal arrow in the lower part. The upward arrows point at relevant properties of the structure and of the behaviour of the network. Relevant properties of the network structure are addressed in Sect. 12.4 (properties of the connectivity structure based on the network's strongly connected components) and Sect. 12.5 (properties of the aggregation structure based on combination functions). For behaviour, in particular, the equilibria that occur will be discussed. Section 12.3 presents basic definitions and shows examples of this. In Sect. 12.6, the main results are presented as depicted

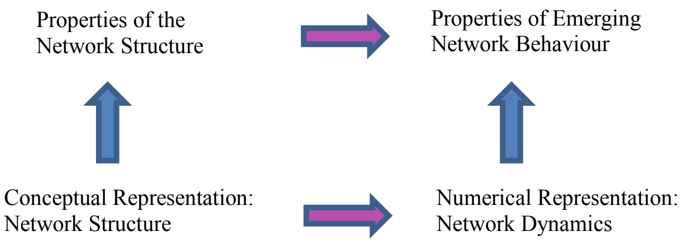

Fig. 12.1 Bottom layer: the conceptual representation defines the numerical representation. Top layer: properties of network structure entail properties of emerging network behaviour

by the upper horizontal arrow in Fig. 12.1. These results mostly have the form that certain network structure properties entail certain network behaviour properties.

12.3 Emerging Behaviour of a Network

Behaviour for $t \to \infty$ will be explored by analysing possible equilibria that can occur.

12.3.1 Basics on Stationary Points and Equilibria for Temporal-Causal Networks

Stationary points and equilibria are defined as follows.

Definition 1 (**stationary point and equilibrium**) A state Y has a *stationary point* at t if $\mathbf{d}Y(t)/\mathbf{d}t = 0$.
The network is in *equilibrium* at t if every state Y of the model has a stationary point at t.

Given the specific differential equation format for a temporal-causal network model the following criterion can be found:

Lemma 1 (**Criterion for a stationary point in a temporal-causal network**) Let Y be a state and $X_1, \ldots, X_k$ the states from which state Y gets its incoming connections. Then Y has a stationary point at t if and only if

$$\boldsymbol{\eta}_Y = 0 \quad \text{or} \quad \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) = Y(t)$$

■

12.3.2 An Example Network

As an illustration the example network shown in Fig. 12.2 is used. The role matrices including the connection weights, speed factors, combination function weights, and combination function parameters, and the initial values used are shown in Box 12.1. The simulation for $\Delta t = 0.5$ is shown in Fig. 12.3.

Note that state X_1 has no incoming connections; in the simulation, it has initial value 0.9 and this stays constant at this level due to having speed factor 0. Also, X_5 has 0.9 as an initial value. The other states have an initial value 0. Note that in Sect. 12.6 theorems are presented from which it follows that the initial values of states X_2 to X_4 and X_8 to X_{10} are irrelevant for the emerging behavior as they do not have any effect on the final behaviour; therefore they were initially set at 0 here.

The speed factor of the states X_2 to X_{10} is 0.5. The combination function used is a normalised scaled sum function (see Sect. 12.5 for more details on normalisation).

12.3.3 Simulations for the Example Network

In the simulation shown in Fig. 12.3 states X_1 to X_4 all end up at value 0.9, states X_5 to X_7 all at value 0.3 and states X_8 to X_{10} at different individual values 0.681, 0.490, and 0.389, respectively. Overall, there is some clustering, but also some states get their unique value. It can be observed that these unique values are in between the cluster values. These observations will be confirmed by the mathematical analysis presented later.

Box 12.1 Role matrices and initial values for the example network shown in Fig. 12.2 as simulated in Fig. 12.3

mb base connectivity	1	2
X_1		
X_2	X_1	X_4
X_3	X_2	
X_4	X_1	X_3
X_5	X_6	
X_6	X_7	
X_7	X_5	
X_8	X_3	X_{10}
X_9	X_5	X_8
X_{10}	X_7	X_9

mcw connection weights	1	2
X_1		
X_2	0.8	0.6
X_3	1	
X_4	0.5	0.2
X_5	0.7	
X_6	0.8	
X_7	0.6	
X_8	0.8	0.6
X_9	0.8	0.8
X_{10}	0.8	0.7

ms speed factors	1
X_1	0
X_2	0.5
X_3	0.5
X_4	0.5
X_5	0.5
X_6	0.5
X_7	0.5
X_8	0.5
X_9	0.5
X_{10}	0.5

mcfw combination function weights	1 eucl
X_1	1
X_2	1
X_3	1
X_4	1
X_5	1
X_6	1
X_7	1
X_8	1
X_9	1
X_{10}	1

mcfp function	1 eucl	
parameter	1 n	2 λ
X_1	1	1
X_2	1	1.4
X_3	1	1
X_4	1	0.7
X_5	1	0.7
X_6	1	0.8
X_7	1	0.6
X_8	1	1.4
X_9	1	1.6
X_{10}	1	1.5

iv initial values	1
X_1	0.9
X_2	0
X_3	0
X_4	0
X_5	0.9
X_6	0
X_7	0
X_8	0
X_9	0
X_{10}	0

The combination function used for the simulation in Fig. 12.3 is the first order euclidean function (or scaled sum function), which is linear. It might be believed that this pattern depends on the combination function being linear. However, this is not

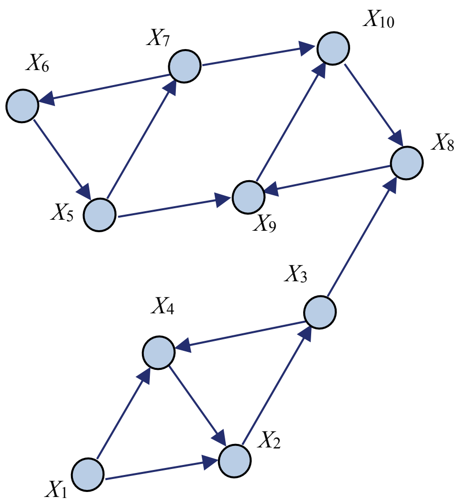

Fig. 12.2 Example network

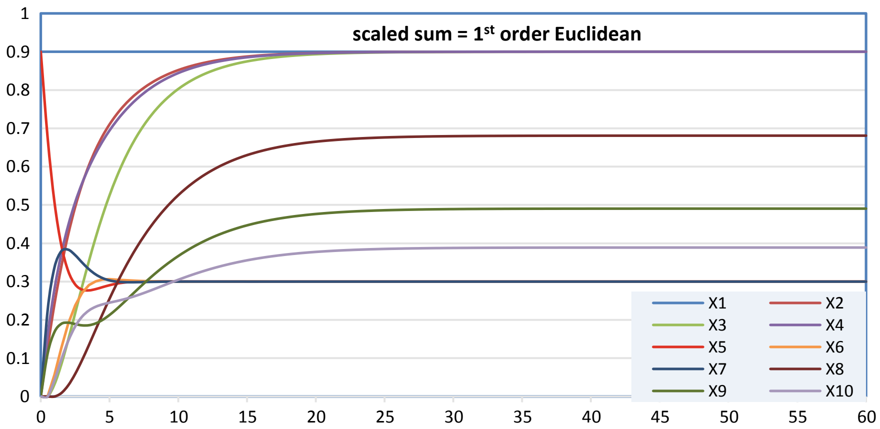

Fig. 12.3 Example simulation for linear scaled sum combination functions

the case. In Fig. 12.4 three simulations are shown for nonlinear combination functions, namely higher order Euclidean combination functions of order 2, 4 and 8, respectively. It is shown that the overall pattern is very similar with the same two groups going for 0.3 and 0.9, and the remaining three states X_8 to X_{10} getting each at different values but between these two values 0.3 and 0.9. The only difference is that the latter three values differ for the four considered combination functions, although they are in the same order. Note that in the graph for the 8th order Euclidean combination function state X_8, in the end, gets a value very close but not equal to 0.9.

The question of how such emerging asymptotic patterns can be explained will be addressed in the next three sections. It will be analysed how the pattern depends on

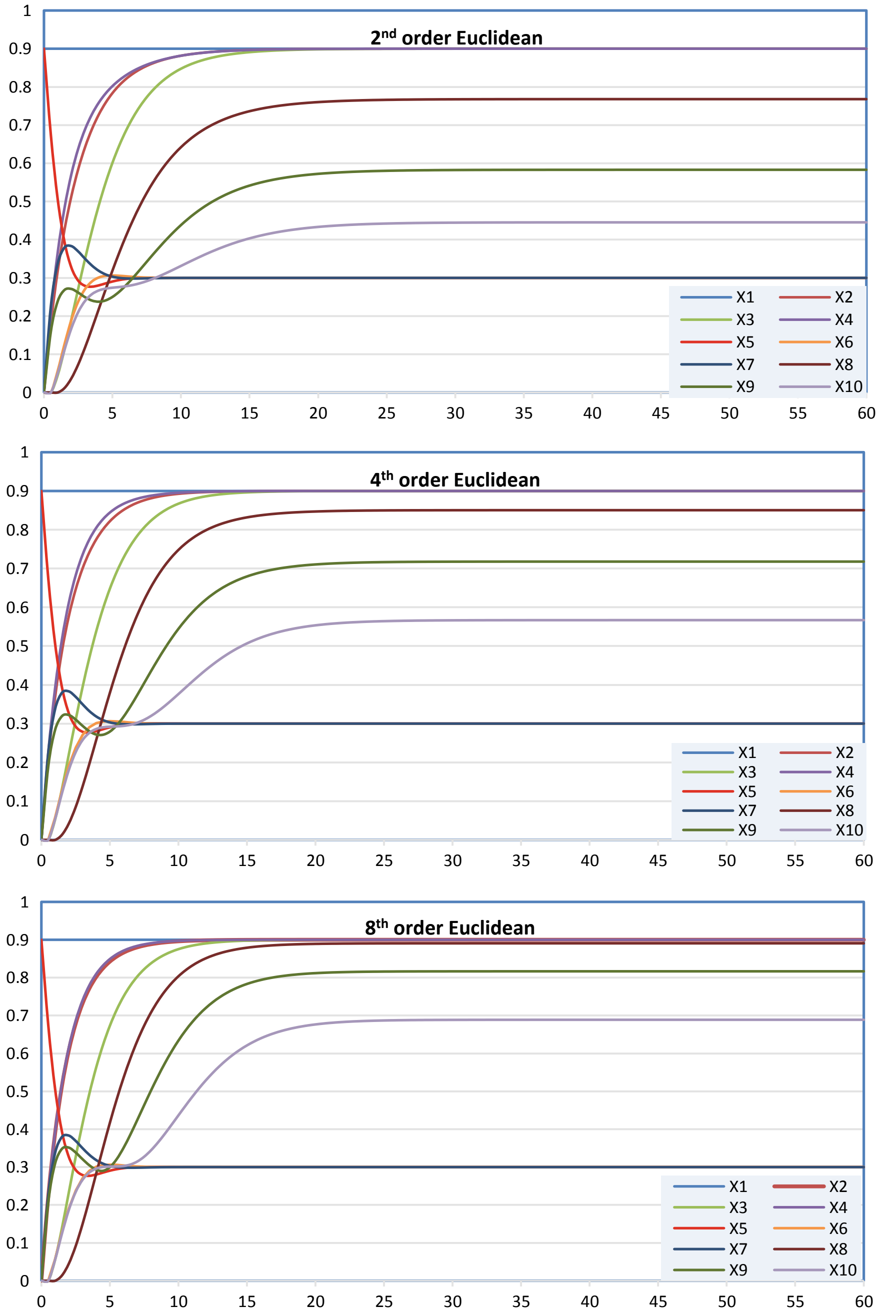

Fig. 12.4 Simulations for nonlinear higher order Euclidean combination functions of order 2, 4, and 8

the network's characteristics, in particular on the connectivity characteristics of the network and the aggregation characteristics modeled by the combination functions. Each of these two factors will be discussed first Sects. 12.4 and 12.5, respectively, after which in Sect. 12.6 they will be related to the emerging behaviour patterns.

12.4 Network Connectivity Characteristics Based on Strongly Connected Components

When broadening the scope of analysis for a wider class of network concerning connectivity characteristics, analysis based on the notion of strongly connected component is useful. Although it had to be rediscovered first, this is known from Graph Theory as turned out afterwards.

12.4.1 A Network's Strongly Connected Components

Most of the following definitions can be found, for example, in (Harary et al. 1965), Chap. 3, or in (Kuich 1970), Sect. 6. Note that here only nonnegative connection weights are considered.

Definition 2 (**reachability and strongly connected components**)

(a) State Y is *reachable* from state X if there is a directed path from X to Y with nonzero connection weights and speed factors.
(b) A network N is *strongly connected* if every two states are mutually reachable within N.
(c) A state is called *independent* if it is not reachable from any other state.
(d) A *subnetwork* of network N is a network whose states and connections are states and connections of N.
(e) A *strongly connected component* C of a network N is a strongly connected subnetwork of N such that no larger strongly connected subnetwork of N contains it as a subnetwork.

Strongly connected components C can be determined by choosing any node X of N and adding all nodes that are on any cycle through X. When a node X is not on any cycle, then it will form a singleton strongly connected component C by itself; this applies to all nodes of N with indegree or outdegree zero. Efficient algorithms have been developed to determine the strongly connected components of a graph; for example, see Bloem et al. (2006), Fleischer et al. (2000), Gentilini et al. (2003), Li et al. (2014), Tarjan (1972), Wijs et al. (2016). The strongly connected components of the example network from Fig. 12.2 are shown in Fig. 12.5.

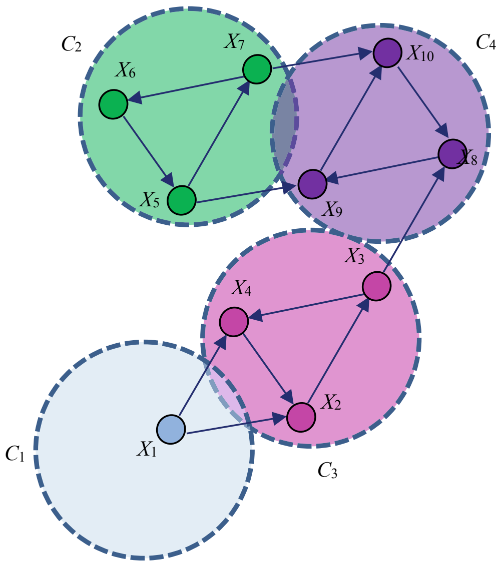

Fig. 12.5 The strongly connected components within the example network

12.4.2 The Stratified Condensation Graph of a Network

Based on the strongly connected components, a form of an abstracted picture of the network can be made, called the condensation graph; see Fig. 12.6.

Definition 3 (**condensation graph**) The *condensation* C(N) of a network N with respect to its strongly connected components is a graph whose nodes are the strongly connected components of N and whose connections are determined as follows: there is a connection from node C_i to node C_j in C(N) if and only if in N there is at least one connection from a node in the strongly connected component C_i to a node in the strongly connected component C_j.

A condensation graph C(N) is always an acyclic graph. The following theorem summarizes this; see also Harary et al. (1965), Chap. 3, Theorems 3.6 and 3.8, or Kuich (1970), Sect. 6.

Theorem 1 (**acyclic condensation graph**)

(a) For any network N, its condensation graph C(N) is acyclic and has at least one state of outdegree zero and at least one state of indegree zero.
(b) The network N is acyclic itself if and only if it is graph-isomorphic to C(N). In this case, the nodes in C(N) all are singleton sets $\{X\}$ containing one state X from N.
(c) The network N is strongly connected itself if and only if C(N) only has one node; this node is the set of all states of N.

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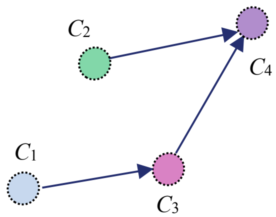

Fig. 12.6 Condensation of the example network by its strongly connected components: the directed acyclic condensation graph C(N)

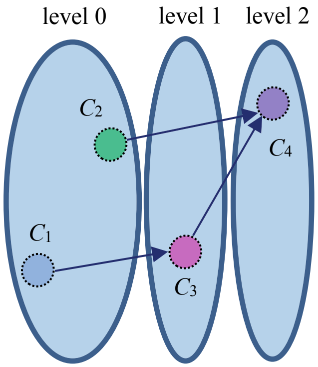

Fig. 12.7 Stratified condensation graph SC(N) for the example network

The structure of an acyclic graph is much simpler than the structure of a cyclic graph. For example, for any acyclic directed graph a stratification structure can be defined; e.g., Chen (2009). Here such construction is applied in particular to the condensation graph C(N) thus obtaining a stratified condensation graph SC (N) which will turn out very useful in Sect. 12.6; see Fig. 12.7.

Definition 4 (**stratified condensation graph**) The *stratified condensation graph* for network N, denoted by SC(N), is the condensation graph C(N) together with a leveled partition $S_0, \ldots, S_{h-1}$ in strata S_i such that $S_0 \cup \ldots \cup S_{h-1}$ is the set of all nodes of C(N) and the S_i are mutually disjoint, which is defined inductively as follows. Here, h is the height of C(N), i.e., the length of the longest path in C(N).

(i) The stratum S_0 is the set of nodes in C(N) without incoming connections in C(N).
(ii) For each $i > 0$ the stratum S_i is the set of nodes in C(N) for which all incoming connections in C(N) come only from nodes in $S_0, \ldots, S_{i-1}$.

If node X is in stratum S_i, its *level* is i.

12.5 Network Aggregation Characteristics Based on Properties of Combination Functions

The following network aggregation characteristics based on properties of combination functions have been found to relate to emerging behaviour as discussed in Sect. 12.3. Note that for combination functions it is (silently) assumed that c($V_1, \ldots, V_k$) = 0 iff $V_i = 0$ for all i.

Definition 5 (monotonic, scalar-free, and additive for a combination function)

(a) A function c(..) is called *monotonically increasing* if for all values U_i, V_i it holds

$$U_i \leq V_i \quad \text{for all} \quad i \Rightarrow \mathrm{c}(U_1, \ldots, U_k) \leq \mathrm{c}(V_1, \ldots, V_k)$$

(b) A function c(..) is called *strictly monotonically increasing* if

$$U_i \leq V_i \quad \text{for all} \quad i, \text{and } U_j < V_j \quad \text{for at least one} \quad j \Rightarrow \mathrm{c}(U_1, \ldots, U_k) < \mathrm{c}(V_1, \ldots, V_k)$$

(c) A function c(..) is called *scalar-free* if for all $\boldsymbol{\alpha} > 0$ and all $V_1, \ldots, V_k$ it holds

$$\mathrm{c}(\boldsymbol{\alpha} V_1, \ldots, \boldsymbol{\alpha} V_k) = \boldsymbol{\alpha}\, \mathrm{c}(V_1, \ldots, V_k)$$

(d) A function c(..) is called *additive* if for all $U_1, \ldots, U_k$ and $V_1, \ldots, V_k$ it holds

$$\mathrm{c}(U_1 + V_1, \ldots, U_k + V_k) = \mathrm{c}(U_1, \ldots, U_k) + \mathrm{c}(V_1, \ldots, V_k)$$

(e) A function c(..) is called *linear* if it is both scalar-free and additive.

Note that these characteristics vary over the different examples of combination functions. Table 12.2 shows which of these characteristics apply to which combination functions. In general, the theorems that follow in Sect. 12.6 have the characteristics (a), (b) and (c) as conditions, so as can be seen in Table 12.2 they apply to **id(.)**, **ssum**$_\lambda$**(..)**, **eucl**$_{n,\lambda}$**(..)**, and **sgeomean**$_\lambda$**(..)** (of which only the first two are linear and the last two are nonlinear, assuming $n \neq 1$ for the third one and nonzero values for the fourth one). The theorems do not apply to **smin**$_\lambda$**(..)** and **smax**$_\lambda$**(..)** (not strictly monotonous), and to **slogistic**$_{\sigma,\tau}$**(..)**, and **alogistic**$_{\sigma,\tau}$**(..)** (not scalar-free). Note that different functions satisfying (a), (b) and (c) can also be combined to get more complex functions by using linear combinations with positive coefficients and function composition.

Table 12.2 Characteristics of Definition 5 for the example combination functions

	(a)	(b)	(c)	(d)	(e)
id(.)	+	+	+	+	+
ssum$_\lambda$**(..)** (= **eucl**$_{n,\lambda}$**(..)** for $n = 1$)	+	+	+	+	+
eucl$_{n,\lambda}$**(..)** for $n \neq 1$	+	+	+	−	−
sgeomean$_\lambda$**(..)** for nonzero values	+	+	+	−	−
smin$_\lambda$**(..)**	+	−	+	−	−
smax$_\lambda$**(..)**	+	−	+	−	−
slogistic$_{\sigma,\tau}$**(..)**	+	+	−	−	−
alogistic$_{\sigma,\tau}$**(..)**	+	+	−	−	−

Definition 6 (**normalised**) A network is *normalised* if for each state Y it holds $c_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) = 1$, where $X_1, \ldots, X_k$ are the states from which Y gets its incoming connections.

As an example, for a Euclidean combination function of nth order the scaling parameter choice $\boldsymbol{\lambda}_Y = \boldsymbol{\omega}_{X_1,Y^n} + \cdots + \boldsymbol{\omega}_{X_k,Y^n}$ will provide a normalised network. This can be done in general as follows:

(1) **normalising a combination function**
If any combination function $\mathbf{c}_Y(..)$ is replaced by $\mathbf{c}'_Y(..)$ defined as

$$\mathbf{c}'_Y(V_1, \ldots, V_k) = \mathbf{c}_Y(V_1, \ldots, V_k)/\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})$$

(note $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}) > 0$ since $\boldsymbol{\omega}_{X_i,Y} > 0$), then the network becomes normalised.

(2) **normalising the connection weights (for scalar-free combination functions)**
For scalar-free combination functions also normalisation is possible by adapting the connection weights; define $\boldsymbol{\omega}'_{X_i,Y} = \boldsymbol{\omega}_{X_i,Y} / \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})$, then indeed it holds:

$$\begin{aligned}\mathbf{c}_Y(\boldsymbol{\omega}'_{X_1,Y}, \ldots, \boldsymbol{\omega}'_{X_k,Y}) &= \mathbf{c}(\boldsymbol{\omega}_{X_1,Y}/c_Y(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y}), \ldots, \boldsymbol{\omega}_{X_k,Y}/ \\ &\qquad \mathrm{c}(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})) = 1\end{aligned}$$

Normalisation is a necessary condition for applying the theorems developed in Sect. 12.6. Simulation is still possible when the network is not normalised. But the effect then usually is that activation is lost in an artificial manner (if the function values are lower than normalised) so that all values go to 0, or that activation is amplified in an artificial manner (if the function values are higher than normalised) so that all values go to 1. That makes less interesting behaviour for practical applications and also less interesting analysis.

For different example functions, following normalisation step (1) above, their normalised variants are given by Table 12.3.

Some of the implications of the above-defined characteristics are illustrated in the following proposition. This will be used in Sect. 12.6; for a proof, see Chap. 15, Sect. 15.7.

Proposition 1 Suppose the network is normalised.

(a) If the combination functions are scalar-free and $X_1, \ldots, X_k$ are the states from which Y gets its incoming connections, and $X_1(t) = \cdots = X_k(t) = V$ for some common value V, then also $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) = V$.

(b) If the combination functions are scalar-free and $X_1, \ldots, X_k$ are the states with outgoing connections to Y, and for $U_1, \ldots, U_k, V_1, \ldots, V_k$ and $\boldsymbol{\alpha} \geq 0$ it holds $V_i = \boldsymbol{\alpha} U_i$, then $\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V_1, \ldots, \boldsymbol{\omega}_{X_k,Y}V_k) = \boldsymbol{\alpha}\, \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}U_1, \ldots, \boldsymbol{\omega}_{X_k,Y}U_k)$.
If in this situation in two different simulations, state values $X_i(t)$ and $X'_i(t)$ are generated then $X'_i(t) = \boldsymbol{\alpha} X_i(t) \quad \Rightarrow X'_i(t + \Delta t) = \boldsymbol{\alpha} X_i(t + \Delta t)$.

Table 12.3 Normalisation of the different examples of combination functions

Combination function	Notation	Normalising scaling factor	Normalised combination function
Identity function	**id(.)**	$\boldsymbol{\omega}_{X,Y}$	$V/\boldsymbol{\omega}_{X,Y}$
Scaled sum	$\mathbf{ssum}_{\lambda}(V_1, \ldots, V_k)$	$\boldsymbol{\omega}_{X_1,Y} + \cdots + \boldsymbol{\omega}_{X_k,Y}$	$(V_1 + \cdots + V_k)/(\boldsymbol{\omega}_{X_1,Y} + \cdots + \boldsymbol{\omega}_{X_k,Y})$
Scaled maximum	$\mathbf{smax}_{\lambda}(V_1, \ldots, V_k)$	$\max(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})$	$\max(V_1, \ldots, V_k)/\max(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})$
Scaled minimum	$\mathbf{smin}_{\lambda}(V_1, \ldots, V_k)$	$\min(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_1,Y})$	$\min(V_1, \ldots, V_k)/\min(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})$
Euclidean	$\mathbf{eucl}_{n,\lambda}(V_1, \ldots, V_k)$	$\boldsymbol{\omega}^n_{X_1,Y} + \cdots + \boldsymbol{\omega}^n_{X_k,Y}$	$\sqrt[n]{\frac{V_1^n + \cdots + V_k^n}{\boldsymbol{\omega}^n_{X_1,Y} + \cdots + \boldsymbol{\omega}^n_{X_k,Y}}}$
Simple logistic	$\mathbf{slogistic}_{\sigma,\tau}(V_1, \ldots, V_k)$	$\mathbf{slogistic}_{\sigma,\tau}(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})$	$\frac{1+e^{-\sigma\omega\left(X_{1,Y}+\cdots+X_{k,Y}-\tau\omega\right)}}{1+e^{-\sigma\left(V_1+\cdots+V_k-\tau\right)}}$
Advanced logistic	$\mathbf{alogistic}_{\sigma,\tau}(V_1, \ldots, V_k)$	$\mathbf{alogistic}_{\sigma,\tau}(\boldsymbol{\omega}_{X_1,Y}, \ldots, \boldsymbol{\omega}_{X_k,Y})$	$\frac{\frac{1}{1+e^{-\sigma\left(V_1+\cdots+V_k-\tau\right)}}-\frac{1}{1+e^{\sigma\tau}}}{\frac{1}{1+e^{-\sigma\omega\left(X_{1,Y}+\cdots+X_{k,Y}-\tau\omega\right)}}-\frac{1}{1+e^{\sigma\tau}}}$

(c) If the combination functions are additive and $X_1, \ldots, X_k$ are the states from which Y gets its incoming connections, then for values $U_1, \ldots, U_k, V_1, \ldots, V_k$ it holds

$$\begin{aligned}\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}(U_1+V_1), \ldots, \boldsymbol{\omega}_{X_k,Y}(U_k+V_k)) &= \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}U_1, \ldots, \boldsymbol{\omega}_{X_k,Y}U_k) \\ &\quad + \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}V_1, \ldots, \boldsymbol{\omega}_{X_k,Y}V_k)\end{aligned}$$

If in this situation in three different simulations, state values $X_i(t)$, $X'_i(t)$ and $X'_i(t)$ are generated then

$$X''_i(t) = X_i(t) + X'_i(t) \quad \Rightarrow \quad X''_i(t+\Delta t) = X_i(t+\Delta t) + X'_i(t+\Delta t)$$

(d) If the combination functions are scalar-free and monotonically increasing, and $X_1, \ldots, X_k$ are the states from which Y gets its incoming connections, and $V_1 \leq X_1(t), \ldots, X_k(t) \leq V_2$ for some values V_1 and V_2, then also

$$V_1 \leq \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) \leq V_2$$

and if $\boldsymbol{\eta}_Y\, \Delta t \leq 1$ and $V_1 \leq Y(t) \leq V_2$ then $V_1 \leq Y(t + \Delta t) \leq V_2$.

12.6 Network Behaviour and Network Structure Characteristics

How the network structure characteristics concerning connectivity and aggregation as discussed in Sects. 12.4 and 12.5 relate to emerging network behaviour is discussed in this section.

12.6.1 Network Behaviour for Special Cases

As a first case, a network without cycles is considered. The following theorem has been proven using Lemma 1 from Sect. 12.3 and Proposition 1; see also Chap. 11, Theorem 1, or Treur (2018a).

Theorem 2 (**common state values provide equilibria**) Suppose a network with nonnegative connections is based on normalised and scalar-free combination functions, and the states without any incoming connection have a constant value. Then the following holds.

(a) Whenever all states have the same value V, the network is in an equilibrium state.
(b) If for every state for its initial value V it holds $V_1 \leq V \leq V_2$, then for all t for every state Y it holds $V_1 \leq Y(t) \leq V_2$. In an achieved equilibrium for every state for its equilibrium value V it holds $V_1 \leq V \leq V_2$.

■

Also this theorem is adopted from Chap. 11, Theorem 2.

Theorem 3 (**Common equilibrium state values; acyclic case**) Suppose an acyclic network with nonnegative connections is based on normalised and scalar-free combination functions.

(a) If in an equilibrium state the independent states all have the same value V, then all states have the same value V.
(b) If, moreover, the combination functions are monotonically increasing, and in an equilibrium state the independent states all have values V with $V_1 \leq V \leq V_2$, then all states have values V with $V_1 \leq V \leq V_2$.

■

The following is a useful basic lemma for dynamics of normalised networks with combination functions that are (strictly) monotonically increasing and scalar-free.

Lemma 2 Let a normalised network with nonnegative connections be given and its combination functions are monotonically increasing and scalar-free; then the following hold:

(a)

  (i) If for some node Y at time t for all nodes X with $\boldsymbol{\omega}_{X,Y} > 0$ it holds $X(t) \leq Y(t)$, then $Y(t)$ is decreasing at t: $\mathbf{d}Y(t)/\mathbf{d}t \leq 0$.
  (ii) If the combination functions are strictly increasing and at time t for all nodes X with $\boldsymbol{\omega}_{X,Y} > 0$ it holds $X(t) \leq Y(t)$, and a node X exists with $X(t) < Y(t)$ and $\boldsymbol{\omega}_{X,Y} > 0$, and the speed factor of Y is nonzero, then $Y(t)$ is strictly decreasing at t: $\mathbf{d}Y(t)/\mathbf{d}t < 0$.

(b)

(i) If for some node Y at time t for all nodes X with $\boldsymbol{\omega}_{X,Y} > 0$ it holds $X(t) \geq Y(t)$, then $Y(t)$ is increasing at t: $\mathbf{d}Y(t)/\mathbf{d}t \geq 0$.

(ii) If, the combination function is strictly increasing and at time t for all nodes X with $\boldsymbol{\omega}_{X,Y} > 0$ it holds $X(t) \geq Y(t)$, and a node X exists with $X(t) > Y(t)$ and $\boldsymbol{\omega}_{X,Y} > 0$, and the speed factor of Y is nonzero, then $Y(t)$ is strictly increasing at t: $\mathbf{d}Y(t)/\mathbf{d}t > 0$.

■

The following theorem has been proven for strongly connected networks with cycles using Lemma 1 and 2; see Chap. 11 (Lemma 2 and Theorem 3) or Treur (2018a).

Theorem 4 (**Common equilibrium state values; strongly connected cyclic case**) Suppose the combination functions of the normalised network N are scalar-free and strictly monotonically increasing. Then the following hold.

(a) If the network is strongly connected itself, then in an equilibrium state all states have the same value.

(b) Suppose the network has one or more independent states and the subnetwork without these independent states is strongly connected. If in an equilibrium state all independent states have values V with $V_1 \leq V \leq V_2$, then all states have values V with $V_1 \leq V \leq V_2$. In particular, when all independent states have the same value V, then all states have this same equilibrium value V.

■

12.6.2 *Network Behaviour for the General Case*

The first general, main theorem is formulated by Theorems 5 and 6.

Theorem 5 (**main theorem on equilibrium state values, part I**) Suppose the network N is normalised and its combination functions are scalar-free and strictly monotonic. Let SC(N) be the stratified condensation graph of N. *Then in an equilibrium state of N the following hold.*

(a) Suppose $C \in$ SC(N) is a strongly connected component of N of level 0, and in case it consists of a single state without any incoming connection, this state has a constant value. Then the following hold:

(i) All states in N belonging to C have the same equilibrium value V.

(ii) If for the initial values V of all states in N belonging to C it holds $V_1 \leq V \leq V_2$, then also for the equilibrium values V of all states in C it holds $V_1 \leq V \leq V_2$.

(iii) In particular, when all initial values of states in N belonging to C are equal to one value V, then the equilibrium value of all states in C is also V.

(b) Let $C \in \mathrm{SC}(N)$ be a strongly connected component of N of level i > 0. Let $C_1, \ldots, C_k \in \mathrm{SC}(N)$ be the strongly connected components of N from which C gets an incoming connection within the condensation graph SC(N). Then the following hold.

(i) If for the equilibrium values V of all states in N belonging to $C_1 \cup \cdots \cup C_k$ it holds $V_1 \leq V \leq V_2$, then for all states in N belonging to C for their equilibrium value V it holds $V_1 \leq V \leq V_2$.

(ii) In particular, when all equilibrium values of all states in N belonging to $C_1 \cup \cdots \cup C_k$ are equal to one value V, then also the equilibrium values of all states in N belonging to C are equal to the same V.

Proof

(a)

(i) follows from Theorem 3(a).
(ii) follows from Proposition 1(b).
(iii) This follows from (ii) with $V_1 = V_2 = V$.

(b)

(i) This follows from Theorem 3(b) applied to C augmented with (as independent states) the states in $C_1 \cup \ldots \cup C_k$ with outgoing connections to states in C, with their values and these connections.
(ii) follows from (i) with $V_1 = V_2 = V$.

■

Theorem 6 (**main theorem on equilibrium state values, part II**) Suppose the network N is normalised and its combination functions are scalar-free and strictly monotonic. Let SC(N) be the stratified condensation graph of N. Then in an equilibrium state of N the following hold.

(a) If the equilibrium values of all states in every strongly connected component of level 0 in SC(N) are equal to one value V, then the equilibrium state values of all states in N are equal to the same value V.
(b) If for the equilibrium values V of all states in every strongly connected component of level 0 in SC(N) it holds $V_1 \leq V \leq V_2$, then for the equilibrium state values V of all states in N it holds $V_1 \leq V \leq V_2$.
(c) Suppose the states without any incoming connection have a constant value. If the initial values of all states in every strongly connected component of level 0 in SC(N) are equal to one value V, then for the equilibrium state values of all states in N are equal to the same value V.

(d) Suppose the states without any incoming connection have a constant value. If for the initial values V of all states in every strongly connected component of level 0 in SC(N) it holds $V_1 \leq V \leq V_2$, then for the equilibrium state values V of all states in N it holds $V_1 \leq V \leq V_2$.

Proof This follows by using induction over the number of strata in SC(N) and applying Theorem 4(a) for the level 0 stratum and Theorem 4(b) for the induction step from the strata of level $j < i$ to the stratum of level $i > 0$.

■

As an illustration, for the example simulation, the following implications of these theorems can be found.

- **Level 0 components**
  The strongly connected components of level 0 are the subnetworks based on $\{X_1\}$ and $\{X_5, X_6, X_7\}$ (see Figs. 12.5 and 12.6). As shown in Box 12.1, the initial values of X_1 and X_5 are 0.9, and the initial values for all other states are 0. From Theorem 5(a)(i) and 4(a)(ii), it follows that the equilibrium value of X_1 is 0.9, which indeed is the case, and those of X_5, X_6, X_7 are the same and $\leq$ 0.9; this is indeed confirmed in Fig. 12.3, as these three equilibrium values of X_5, X_6, X_7 are all 0.3. This value 0.3 depends on the initial values of the states and the connection weights, which are not taken into account in the theorems; however, see also Theorem 7 below.
- **Level 1 component**
  For the level 1 component C_3, based on $\{X_2, X_3, X_4\}$, it goes as follows. The only incoming connection for C_3 is from X_1, which has equilibrium value 0.9 (implied by Theorem 5(a)(ii)). By Theorem 5(b)(ii) it follows that X_2, X_3, X_4 all have the same equilibrium value 0.9; this is indeed confirmed in Fig. 12.3.
- **Level 2 component**
  The level 2 component C_4 is based on $\{X_8, X_9, X_{10}\}$. It has two incoming connections, one from X_3 in C_3 and one from X_5 in C_2. Their equilibrium values are 0.9 and 0.3, respectively, so they are not equal. Therefore the above theorems do not imply that the equilibrium values of X_8, X_9, X_{10} are the same; indeed in Fig. 12.3 they are different: 0.681, 0.490, and 0.389, respectively. But there is still an implication from Theorem 5(b)(i), namely, that these equilibrium values should be $\geq$ 0.3 and $\leq$ 0.9. This is indeed confirmed in Fig. 12.3.

This illustrates how the above theorems have implications for simulations. Note that the specific equilibrium values 0.681, 0.490, and 0.389 are not predicted here. They also depend on the connection weights for the states X_8, X_9, X_{10} within component C_4, and these are not taken into account in the theorems; however see also below, in the last part of this section.

Consider a variation, by setting the initial value of X_1 at 0.3 instead of 0.9. Then all equilibrium values turn out to become the same 0.3; see Fig. 12.8. Now the values of all states in the level 0 components C_1 and C_2 have the same value 0.3. As above, also the states in C_3 have the equilibrium value 0.3 because they are only

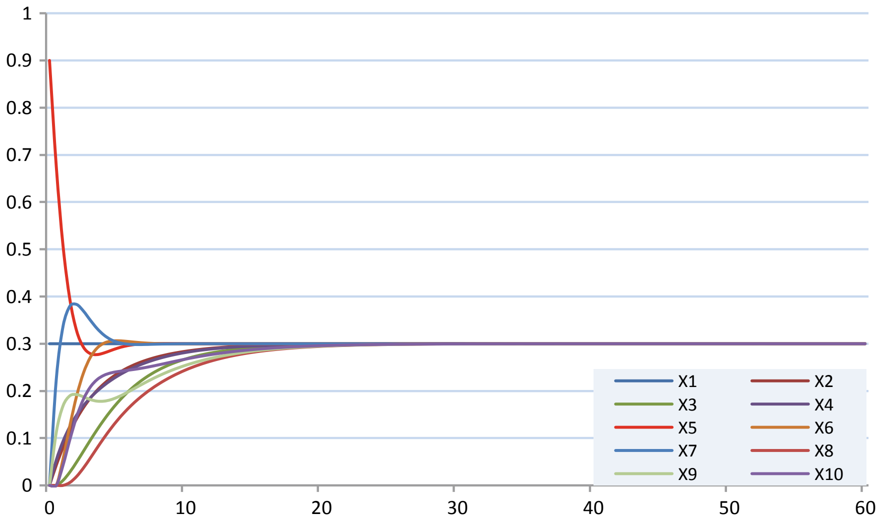

Fig. 12.8 Variation of the example simulation for initial value 0.3 of X_1

affected by X_1 which has value 0.3. But now the equilibrium values of both X_3 in C_3 and X_5 in C_2 are the same 0.3, so this time Theorem 5(b)(ii) can be applied to derive that all states in C_4 also have that same equilibrium value 0.3.

This predicts that all states of the network have value 0.3 in the equilibrium. Alternatively, Theorem 6(a) can be applied for this case. By that theorem from the equal equilibrium values in the level 0 components C_1 and C_2 it immediately follows that all states in all components in the network have that same equilibrium value.

As seen above, in the theorems the level 0 components play a central role, as initial nodes in the stratified condensation graph SC(N). Therefore it can be useful to know more about them, for example, how their initial values determine all equilibrium values in the network. This is addressed for the case of a linear combination function in the following theorem. For a proof, see Chap. 15, Sect. 15.7.

Theorem 7 (**equilibrium state values in relation to level 0 components in the linear case**) Suppose the network N is normalised and the combination functions are strictly monotonically increasing and linear. Assume that the states at level 0 that form a singleton component on their own are constant.
Then the following hold:

(a) For each state Y its equilibrium value is independent of the initial values of all states at some level $i > 0$. It is only dependent on the initial values for the states at level 0.

(b) More specifically, let $B_1, \ldots, B_p$ be the states in level 0 components. Then for each state Y its equilibrium value eq_Y is described by a linear function of the initial values $V_1, \ldots, V_p$ for $B_1, \ldots, B_p$, according to the following weighted average:

$$\mathrm{eq}_Y\left(V_1, \ldots, V_p\right) = d_{B_1,Y} V_1 + \cdots + d_{B_p,Y} V_p$$

Here the $d_{B_i,Y}$ are real numbers between 0 and 1 and the sum of them is 1:

$$d_{B_1,Y} + \cdots + d_{B_p,Y} = 1$$

(c) Each $d_{B_i,Y}$ is the equilibrium value for Y when the following initial values are used: $V_i = 1$ and all other initial values are 0:

$$d_{B_i,Y} = \mathrm{eq}_Y(0, \ldots, 0, 1, 0, \ldots, 0) \text{ with 1 as } i\text{th argument.}$$

Note that Theorem 7(c) can be used to determine the values of the numbers $d_{B_i,Y}$ by simulation for each of these p initial value settings. However, in Sect. 12.7 it will also be shown how they can be determined by symbolically solving the equilibrium equations. Based on Theorem 7, for the case of linear combination functions, for level 0 components after each value $d_{B_i,Y}$ is determined, any equilibrium value can be predicted from the initial values by the identified linear expression.

Note that for the case of linear combination functions the equilibrium equations are linear and could be solved algebraically. But this does not provide additional information for nonsingleton level 0 components. They have an infinite number of solutions as every common value V is a solution; apparently, the linear equations always have a mutual dependency in this case. However, for components of level $i > 0$, solving the linear equations can provide specific values, due to the specific input values they get from one or more lower level components. In Sect. 12.7 such implications of the theorems for some example networks are shown. The next theorems show some variations on Theorem 7.

Theorem 8 (equilibrium state values for level 0 components) Suppose the network N with states $X_1, \ldots, X_n$ is normalised and strongly connected. Then the following hold.

(a) If the combination functions of the network N are scalar-free, then for given connection weights and speed factors, for any value $V \in [0, 1]$ there are initial values such that V is the common state value in an equilibrium achieved from these initial values.

(b) For given connection weights and speed factors, let eq: $[0, 1]^n \to [0, 1]$ be the function such that $\mathrm{eq}(V_1, \ldots, V_n)$ is the common state value for an equilibrium achieved from initial values $X_i(0) = V_i$ for all i. Then $\mathrm{eq}(0, \ldots, 0) = 0$, $\mathrm{eq}(1,.., 1) = 1$, and the following hold:

 (i) If the combination functions of the network are scalar-free, then eq is scalar-free

(ii) If the combination functions of the network are additive, then eq is additive.

(c) Suppose the combination functions of the network N are linear. For given connection weights and speed factors for each i let e_i be the achieved common equilibrium value for initial values $X_i(0) = 1$ and $X_j(0) = 0$ for all $j \neq i$, *i.e.*, $e_i = \text{eq}(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 as ith argument. Then the sum of the e_i *is* 1, *i.e.*, $e_1 + \cdots + e_n = 1$ and in the general case for these given connection weights and speed factors, the common equilibrium value eq(…) is a linear, monotonically increasing, continuous and differentiable function of the initial values $V_1, \ldots, V_n$ satisfying the following linear relation:

$$\text{eq}(V_1, \ldots, V_n) = e_1 V_1 + \ldots + e_n V_n$$

If the combination functions of N are strictly increasing, then $e_i > 0$ for all i, and eq is also strictly increasing.

Proof (a) This follows from Proposition 1(a) or (d) with $V_1 = V_2 = V$.
(b) and (c) This follows from Proposition 1(b) and (c), and Lemma 2.

■

For a proof of the following theorem, see Chap. 15, Sect. 15.7.

Theorem 9 (equilibrium state values for components of level $i > 0$) Suppose the network is normalised, and consists of a strongly connected component plus a number of independent states $A_1, \ldots, A_p$ with outgoing connections to this strongly connected component. Then the following hold

(a) Suppose the combination functions are scalar-free and $X_1, \ldots, X_k$ are the states from which Y gets its incoming connections. If for $U_1, \ldots, U_k, V_1, \ldots, V_k$ and $\boldsymbol{\alpha} \geq 0$ it holds $V_i = \boldsymbol{\alpha} U_i$ for all i, then $c_Y(\boldsymbol{\omega}_{X_1,Y} V_1, \ldots, \boldsymbol{\omega}_{X_k,Y} V_k) = \boldsymbol{\alpha}\, c_Y(\boldsymbol{\omega}_{X_1,Y} U_1, \ldots, \boldsymbol{\omega}_{X_k,Y} U_k)$
(b) Suppose the combination functions are additive and $X_1, \ldots, X_k$ are the states from which Y gets its incoming connections. Then if for values $U_1, \ldots, U_k, V_1, \ldots, V_k, W_1, \ldots, W_k$ it holds $W_i = U_i + V_i$ for all i, then

$$\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} W_1, \ldots, \boldsymbol{\omega}_{X_k,Y} W_k) = \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} U_1, \ldots, \boldsymbol{\omega}_{X_k,Y} U_k) + \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} V_1, \ldots, \boldsymbol{\omega}_{X_k,Y} V_k)$$

(c) Suppose all combination functions of the network N are linear. Then for given connection weights and speed factors, for each state Y the achieved equilibrium value for Y only depends on the equilibrium values $V_1, \ldots, V_p$ of states $A_1, \ldots, A_p$; the function $\text{eq}_Y(V_1, \ldots, V_p)$ denotes this achieved equilibrium value for Y.
(d) Suppose the combination functions of the network N are linear. For the given connection weights and speed factors for each i let $d_{i,Y}$ be the achieved equilibrium value for state Y in a situation with equilibrium values $A_i = 1$ and $A_j = 0$ for all $j \neq i$, i.e., $d_{i,Y} = \text{eq}_Y(0, \ldots, 0, 1, 0,.., 0)$ with 1 as ith argument.

Then in the general case for these given connection weights and speed factors, for each Y in the strongly connected component its equilibrium value is a linear, monotonically increasing, continuous and differentiable function $\mathrm{eq}_Y(\ldots)$ of the equilibrium values $V_1, \ldots, V_p$ of $A_1, \ldots, A_p$ satisfying the following linear relation: $\mathrm{eq}_Y(V_1, \ldots, V_p) = d_{1,Y}\, V_1 + \cdots + d_{p,Y}\, V_p$. Here the sum of the $d_{i,Y}$ *is* 1: $d_{1,Y} + \cdots + d_{p,Y} = 1$. In particular, the equilibrium values are independent of the initial values for all states Y different from $A_1, \ldots, A_p$. If the combination functions of N are strictly increasing, then $d_{i,Y} > 0$ for all i, and $\mathrm{eq}_Y(..)$ is also strictly increasing.

Note that by using Theorem 3 instead of Theorem 5(b)(ii) in the above proof a similar theorem is obtained for the case of an acyclic network: then the equilibrium values of all states are linear combinations of the values of the initial states.

12.7 Further Implications for Example Networks

In this section, it is shown what further conclusions can be drawn from the theorems presented in Sect. 12.6 for the example described in Sect. 12.3 and for an example Mental Network described in Schoenmaker et al. (2018). This shows that the applicability goes beyond only Social Networks. First, the earlier example described in Sect. 12.3 is analysed; after that the new example will be addressed.

12.7.1 Further Analysis of the Example Network from Sect. 12.3.2

Theorems 7 to 9 are illustrated by the example network shown in Fig. 12.2 as follows. Here there is only one independent constant state X_1 with singleton component. Moreover, the states in the other level 0 component C_2 are X_5, X_6, X_7 respectively (see Fig. 12.5). So, from Theorem 7 it follows that the equilibrium value of any state Y is

$$\mathrm{eq}_Y(V_1,\ V_2,\ V_3, V_4) = d_{X_1,Y}V_1 + d_{X_5,Y}V_2 + d_{X_6,Y}V_3 + d_{X_7,Y}V_4 \tag{12.3}$$

where V_1, V_2, V_3, V_4 are the initial values of the states X_1, X_5, X_6, X_7 in the level 0 components C_1 and C_2. For the example states $Y \in \{X_8, X_9, X_{10}\}$ the coefficients $d_{X1,Y}$, $d_{X5,Y}$, $d_{X6,Y}$, $d_{X7,Y}$ have been determined by simulation for the connection weights shown in Box 12.1 (and using speed factors 0.5), with these results shown in Table 12.4.

So, for example, for $Y = X_8$ the four coefficients are:

$$d_{X_1,X_8} = 0.634921 \quad d_{X_5,X_8} = 0.121693 \quad d_{X_6,X_8} = 0.121693 \quad d_{X_7,X_8} = 0.121693$$

Table 12.4 Coefficients of the linear relations between equilibrium values and initial values

	X_1	X_5	X_6	X_7
X_8	0.634921	0.121693	0.121693	0.121693
X_9	0.31746	0.227513	0.227513	0.227513
X_{10}	0.148148	0.283951	0.283951	0.283951
	$d_{X_1,Y}$	$d_{X_5,Y}$	$d_{X_6,Y}$	$d_{X_7,Y}$

Therefore, in a sense, the equilibrium value of X_8 can be considered to be determined for 63.5% by the constant value of X_1 and for 12.2% by each of the initial values of X_5, X_6, X_7. These four values indeed sum up to 1 or 100%. Note that in this case, the last three coefficients happen to be equal, as for the sake of simplicity this component is just one cycle and is therefore highly symmetric; this is not always the case. More specifically, given the above values, for the considered case the equilibrium value for X_8 is given by

$$\mathrm{eq}_{X_8}(V_1, V_2, V_3, V_4) = 0.634921V_1 + 0.121693V_2 + 0.121693V_3 + 0.121693V_4 \tag{12.4}$$

with V_1 the constant value of X_1 and V_2, V_3, V_4 the initial values of X_5, X_6, X_7, respectively. This is indeed confirmed in simulations.

For the considered example, as the scaled sum used is linear, solving the linear equations can also provide specific values. In this way, in line with Theorems 7 to 9 the specific equilibrium values of the states X_8, X_9 and X_{10} in C_4 can be determined algebraically from the values of the states X_3, X_5, and X_7 in the lower level components C_2 and C_3 (repetitive digits in italics). Using a symbolic solver (the online WIMS Linear Solver tool[1] was used), this can be done more in general. The linear equilibrium equations are:

$$\begin{aligned}
(\boldsymbol{\omega}_{X_3,X_8} + \boldsymbol{\omega}_{X_{10},X_8})X_8 &= \boldsymbol{\omega}_{X_3,X_8}X_3 + \boldsymbol{\omega}_{X_{10},X_8}X_{10} \\
(\boldsymbol{\omega}_{X_5,X_9} + \boldsymbol{\omega}_{X_8,X_9})X_9 &= \boldsymbol{\omega}_{X_5,X_9}X_5 + \boldsymbol{\omega}_{X_8,X_9}X_8 \\
(\boldsymbol{\omega}_{X_7,X_{10}} + \boldsymbol{\omega}_{X_9,X_{10}})X_{10} &= \boldsymbol{\omega}_{X_7,X_{10}}X_7 + \boldsymbol{\omega}_{X_9,X_{10}}X_9
\end{aligned} \tag{12.5}$$

These general equations have the following unique symbolic solution when X_3, X_5, X_7 are assumed given from the lower level components:

[1]http://wims.unice.fr/wims/wims.cgi?session=DH1DFC9A6E.3&+lang=en&+module=tool%2Flinear%2Flinsolver.en.

$$\begin{aligned} X_8 = (&\boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9}X_7 + \boldsymbol{\omega}_{X_5,X_9}\boldsymbol{\omega}_{X_7,X_{10}}X_7 + \boldsymbol{\omega}_{X_5,X_9}\boldsymbol{\omega}_{X_9,X_{10}}X_5) \\ &+ \boldsymbol{\omega}_{X_3,X_8}(\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}))X_3) \\ &/(\boldsymbol{\omega}_{X_3,X_8}(\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})) \\ &+ \boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} X_9 = (&\boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9}X_7 + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})X_5) \\ &+ \boldsymbol{\omega}_{X_3,X_8}\boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})X_5 + \boldsymbol{\omega}_{X_3,X_8}\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})X_3) \\ &/(\boldsymbol{\omega}_{X_3,X_8}(\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})) \\ &+ \boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} X_{10} = (&\boldsymbol{\omega}_{X_3,X_8}(\boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9}X_7 + \boldsymbol{\omega}_{X_5,X_9}\boldsymbol{\omega}_{X_7,X_{10}}X_7 + \boldsymbol{\omega}_{X_5,X_9}\boldsymbol{\omega}_{X_9,X_{10}}X_5) \\ &+ \boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_7,X_{10}}w_{X_8,X_9}X_7 + \boldsymbol{\omega}_{X_5,X_9}\boldsymbol{\omega}_{X_7,X_{10}}X_7 + \boldsymbol{\omega}_{X_5,X_9}\boldsymbol{\omega}_{X_9,X_{10}}X_5) \\ &+ \boldsymbol{\omega}_{X_3,X_8}\boldsymbol{\omega}_{X_8,X_9}\boldsymbol{\omega}_{X_9,X_{10}}X_3)/(\boldsymbol{\omega}_{X_3,X_8}(\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})) \\ &+ \boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9})) \end{aligned}$$

From this the values of the coefficients $d_{X_i,Y}$ of the linear relation from Theorem 8 can be determined:

$$\begin{aligned} X_8 = (&\boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9}X_7 + \boldsymbol{\omega}_{X_5,X_9}\boldsymbol{\omega}_{X_7,X_{10}}X_7 + \boldsymbol{\omega}_{X_5,X_9}\boldsymbol{\omega}_{X_9,X_{10}}X_5) \\ &+ \boldsymbol{\omega}_{X_3,X_8}(\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}))X_3) \\ &/(\boldsymbol{\omega}_{X_3,X_8}(\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})) \\ &+ \boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} d_{X_3,X_8} = &\boldsymbol{\omega}_{X_3,X_8}(\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})) \\ &/(\boldsymbol{\omega}_{X_3,X_8}(\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})) \\ &+ \boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} d_{X_5,X_8} = &\boldsymbol{\omega}_{X_{10},X_8}\boldsymbol{\omega}_{X_5,X_9}\boldsymbol{\omega}_{X_9,X_{10}} \\ &/(\boldsymbol{\omega}_{X_3,X_8}(\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})) \\ &+ \boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} d_{X_7,X_8} = &\boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9} + \boldsymbol{\omega}_{X_5,X_9}\boldsymbol{\omega}_{X_7,X_{10}}) \\ &/(\boldsymbol{\omega}_{X_3,X_8}(\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})) \\ &+ \boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} X_9 = (&\boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9}X_7 + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + w_{X_7,X_{10}})X_5) \\ &+ \boldsymbol{\omega}_{X_3,X_8}\boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})X_5 + \boldsymbol{\omega}_{X_3,X_8}\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})X_3) \\ &/(\boldsymbol{\omega}_{X_3,X_8}(\boldsymbol{\omega}_{X_8,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}})) \\ &+ \boldsymbol{\omega}_{X_{10},X_8}(\boldsymbol{\omega}_{X_5,X_9}(\boldsymbol{\omega}_{X_9,X_{10}} + \boldsymbol{\omega}_{X_7,X_{10}}) + \boldsymbol{\omega}_{X_7,X_{10}}\boldsymbol{\omega}_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} d_{X_3,X_9} = {} & \omega_{X_3,X_8}\omega_{X_8,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) \\ & /(w_{X_3,X_8}(w_{X_8,X_9}(w_{X_9,X_{10}} + w_{X_7,X_{10}}) + w_{X_5,X_9}(w_{X_9,X_{10}} + w_{X_7,X_{10}})) \\ & + \omega_{X_{10},X_8}(\omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_7,X_{10}}\omega_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} d_{X_5,X_9} = {} & \omega_{X_5,X_9}(\omega_{X_{10},X_8} + \omega_{X_3,X_8})(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) \\ & /(\omega_{X_3,X_8}(\omega_{X_8,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}})) \\ & + \omega_{X_{10},X_8}(\omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_7,X_{10}}\omega_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} d_{X_7,X_9} = {} & \omega_{X_{10},X_8}\omega_{X_7,X_{10}}\omega_{X_8,X_9} \\ & /(\omega_{X_3,X_8}(\omega_{X_8,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}})) \\ & + \omega_{X_{10},X_8}(\omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_7,X_{10}}\omega_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} X_{10} = {} & (\omega_{X_3,X_8}(\omega_{X_7,X_{10}}\omega_{X_8,X_9}X_7 + \omega_{X_5,X_9}\omega_{X_7,X_{10}}X_7 + \omega_{X_5,X_9}\omega_{X_9,X_{10}}X_5) \\ & + \omega_{X_{10},X_8}(\omega_{X_7,X_{10}}\omega_{X_8,X_9}X_7 + \omega_{X_5,X_9}X_7 + \omega_{X_5,X_9}\omega_{X_7,X_{10}}X_7 + \omega_{X_5,X_9}\omega_{X_9,X_{10}}X_5) \\ & + \omega_{X_3,X_8}\omega_{X_8,X_9}\omega_{X_9,X_{10}}X_3)/(\omega_{X_3,X_8}(\omega_{X_8,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}})) \\ & + \omega_{X_{10},X_8}(\omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_7,X_{10}}\omega_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} d_{X_3,X_{10}} = {} & \omega_{X_3,X_8}\omega_{X_8,X_9}\omega_{X_9,X_{10}}/(\omega_{X_3,X_8}(\omega_{X_8,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}})) \\ & + \omega_{X_{10},X_8}(\omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_7,X_{10}}\omega_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} d_{X_5,X_{10}} = {} & \omega_{X_5,X_9}\omega_{X_9,X_{10}}(\omega_{X_3,X_8} + \omega_{X_{10},X_8}) \\ & /(\omega_{X_3,X_8}(\omega_{X_8,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}})) \\ & + \omega_{X_{10},X_8}(\omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_7,X_{10}}\omega_{X_8,X_9})) \end{aligned}$$

$$\begin{aligned} d_{X_7,X_{10}} = {} & \omega_{X_7,X_{10}}(\omega_{X_3,X_8} + \omega_{X_{10},X_8})(\omega_{X_8,X_9} + \omega_{X_5,X_9}) \\ & /(\omega_{X_3,X_8}(\omega_{X_8,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}})) \\ & + \omega_{X_{10},X_8}(\omega_{X_5,X_9}(\omega_{X_9,X_{10}} + \omega_{X_7,X_{10}}) + \omega_{X_7,X_{10}}\omega_{X_8,X_9})) \end{aligned}$$

As a special case, if all ω's are set equal to one ω, then the denominator becomes $7\omega^3$ and the following values are obtained:

$$\begin{array}{lll} d_{X_3,X_8} = 4/7 & d_{X_3,X_9} = 2/7 & d_{X_3,X_{10}} = 1/7 \\ d_{X_5,X_8} = 1/7 & d_{X_5,X_9} = 4/7 & d_{X_5,X_{10}} = 2/7 \\ d_{X_7,X_8} = 2/7 & d_{X_7,X_9} = 1/7 & d_{X_7,X_{10}} = 4/7 \end{array}$$

Then the linear relations for the equilibrium values become:

$$X_8 = {}^{4}\!/_{7} X_3 + {}^{1}\!/_{7} X_5 + {}^{2}\!/_{7} X_7$$
$$X_9 = {}^{2}\!/_{7} X_3 + {}^{4}\!/_{7} X_5 + {}^{1}\!/_{7} X_7$$
$$X_{10} = {}^{1}\!/_{7} X_3 + {}^{2}\!/_{7} X_5 + {}^{4}\!/_{7} X_7$$

12.8 Analysis of an Example Mental Network

In this section, applicability is illustrated for a type of network which is not a social network. In general Theorems 7 to 9 can be applied for many cases of networks that receive external input. This varies from Mental Networks that get input from external stimuli to Social Networks that are affected by context factors such as broadcasts from external sources that are received by members of the network. As an example of this, for the mental area, the Mental Network model from Schoenmaker et al. (2018) has been analysed. The strongly connected components are as shown in Fig. 12.9, with stratified condensation graph as in Fig. 12.10; for the connection weights and other values, see the role matrices in Box 12.5. The model describes how the emotional charge of a received tweet affects the decision to retweet it. It can be explained by the following scenario considering Mark sending a tweet to Tim in which he expresses that he cannot wait to sing in the Christmas choir next week.

> This tweet contains both information and emotional charge: there is a choir performance next week, and secondly, Mark makes clear that he cannot wait for this event to happen. Tim's interpretation of this message is positively influenced by the fact that Mark and Tim

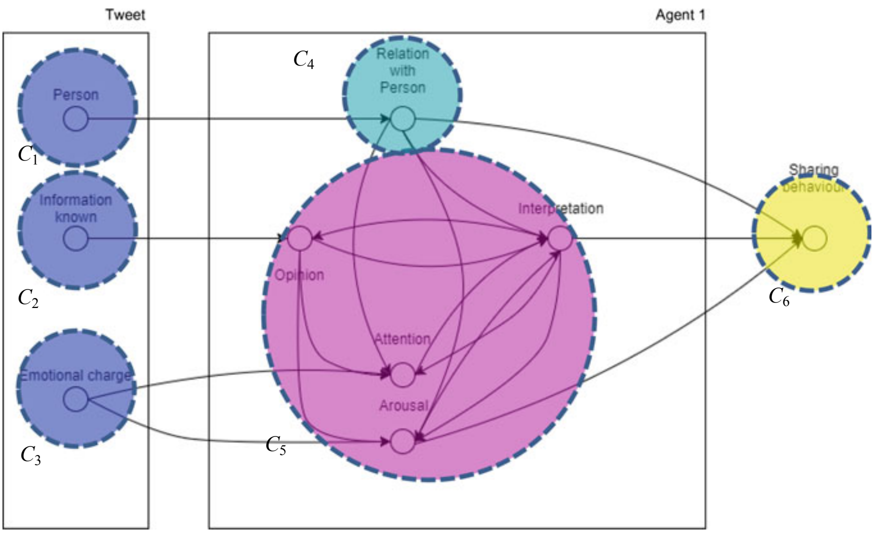

Fig. 12.9 The strongly connected components within the second example network

> are friends. Tim does like to visit choir performances; therefore, he already has a positive association on the information that this event will take place. Reading about this Christmas performance, Tim gets slightly aroused and is focusing on the message. Mark's enthusiasm amplifies Tim's attention and arousal, which in turn lead to a positive interpretation of the tweet. Tim's positive interpretation of the message coupled with the fact that he is good friends with Mark and is excited about this performance leads to Tim's decision to retweet Mark's original Tweet. (Schoenmaker et al. 2018), p. 138

The states within the box Agent 1 all have a scaled sum combination function. The final state Sharing has **alogistic$_{\sigma,\tau}$(..)** as combination function. For a complete overview of the role matrices, see Box 12.5. For the analysis, the above theorems can be applied to the network when the state Sharing is left out of consideration.

The stratified condensation graph for this network is shown in Fig. 12.10. From this stratified condensation graph a number of conclusions can be drawn:

- The level 0 states are the states Person, Information known and Emotional charge in C_1, C_2, and C_3, respectively; therefore these three states are the determining factors for the whole network.
- The level 1 state Relation with person will have the same equilibrium value as the level 0 state Person in C_1.
- When all level 0 states have the same equilibrium value V, then also all level 1 and level 2 states Relation with person, Opinion, Attention, Arousal, and Interpretation will have that same equilibrium value V. For example, when all level 0 states are constant 1, then all states as mentioned will end up in equilibrium value 1.
- When the level 0 states have different equilibrium values, then the level 2 states Opinion, Attention, Arousal and Interpretation are expected to have different equilibrium values too, these values lay between the maximal and minimal values at level 0.

More specifically, in numbers, the following can be concluded. Suppose any given constant values A_1, A_2, A_3 for the level 0 components in C_1, C_2, C_3, respectively. Then:

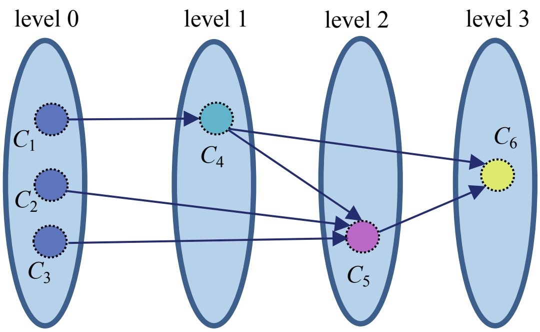

Fig. 12.10 Stratified condensation graph SC(N) for the second example network

- at level 1 the equilibrium value in C_4 is A_1
- at level 2 the equilibrium values of all four states in C_5 are between $\min(A_1, A_2, A_3)$ and $\max(A_1, A_2, A_3)$
- these equilibrium values of the four states in C_5 are linear functions in the form of weighted sums of A_1, A_2, A_3
- when all $A_i = A$ for one value, then at level 2 the equilibrium values of the states in C_5 are A as well

In Box 12.2 the differential equations are shown for this second example model, and in Box 12.3 the equilibrium equations.

Box 12.2 Overview of the differential equations of the second example network models

$$
\begin{aligned}
\mathbf{d}\text{Relation}/\mathbf{d}\text{t} &= \boldsymbol{\eta}_{\text{Relation}}[\boldsymbol{\omega}_{\text{Person,Relation}}\text{Person} - \text{Relation}] \\
\mathbf{d}\text{Opinion}/\mathbf{d}\text{t} &= \boldsymbol{\eta}_{\text{Opinion}}[(\boldsymbol{\omega}_{\text{Information,Opinion}}\text{Information} \\
&\quad + \boldsymbol{\omega}_{\text{Interpretation,Opinion}}\text{Interpretation})/\boldsymbol{\lambda}_{\text{Opinion}} - \text{Opinion}] \\
\mathbf{d}\text{Interpretation}/\mathbf{d}\text{t} &= \boldsymbol{\eta}_{\text{Interpretation}}[(\boldsymbol{\omega}_{\text{Relation,Interpretation}}\text{Relation} \\
&\quad + \boldsymbol{\omega}_{\text{Opinion,Interpretation}}\text{Opinion} \\
&\quad + \boldsymbol{\omega}_{\text{Attention,Interpretation}}\text{Attention} \\
&\quad + \boldsymbol{\omega}_{\text{Arousal,Interpretation}}\text{Arousal})/\boldsymbol{\lambda}_{\text{Interpretation}} - \text{Interpretation}] \\
\mathbf{d}\text{Attention}/\mathbf{d}\text{t} &= \boldsymbol{\eta}_{\text{Attention}}[(\boldsymbol{\omega}_{\text{Emotion,Attention}}\text{Emotion} + \boldsymbol{\omega}_{\text{Relation,Attention}}\text{Relation} \\
&\quad + \boldsymbol{\omega}_{\text{Opinion,Attention}}\text{Opinion} \\
&\quad + \boldsymbol{\omega}_{\text{Interpretation,Attention}}\text{Interpretation})/\boldsymbol{\lambda}_{\text{Attention}} - \text{Attention}] \\
\mathbf{d}\text{Arousal}/\mathbf{d}\text{t} &= \boldsymbol{\eta}_{\text{Arousal}}[(\boldsymbol{\omega}_{\text{Emotion,Arousal}}\text{Emotion} + \boldsymbol{\omega}_{\text{Relation,Arousal}}\text{Relation} \\
&\quad + \boldsymbol{\omega}_{\text{Opinion,Arousal}}\text{Opinion} \\
&\quad + \boldsymbol{\omega}_{\text{Interpretation,Arousal}}\text{Interpretation})/\boldsymbol{\lambda}_{\text{Arousal}} - \text{Arousal}] \\
\mathbf{d}\text{Sharing}/\mathbf{d}\text{t} &= \boldsymbol{\eta}_{\text{Sharing}}[(\boldsymbol{\omega}_{\text{Relation,Sharing}}\text{Relation} \\
&\quad + \boldsymbol{\omega}_{\text{Interpretation,Sharing}}\text{Interpretation} \\
&\quad + \boldsymbol{\omega}_{\text{Arousal,Sharing}}\text{Arousal})/\boldsymbol{\lambda}_{\text{Sharing}} - \text{Sharing}]
\end{aligned}
$$

Box 12.3 Overview of the equilibrium equations of the second example network model

$$
\begin{aligned}
\text{Relation} &= \boldsymbol{\omega}_{\text{Person,Relation}}\text{Person}\\
\text{Opinion} &= (\boldsymbol{\omega}_{\text{Information,Opinion}}\text{Information}\\
&\quad + \boldsymbol{\omega}_{\text{Interpretation,Opinion}}\text{Interpretation})/\boldsymbol{\lambda}_{\text{Opinion}}\\
\text{Interpretation} &= (\boldsymbol{\omega}_{\text{Relation,Interpretation}}\text{Relation} + \boldsymbol{\omega}_{\text{Opinion,Interpretation}}\text{Opinion}\\
&\quad + \boldsymbol{\omega}_{\text{Attention,Interpretation}}\text{Attention}\\
&\quad + \boldsymbol{\omega}_{\text{Arousal,Interpretation}}\text{Arousal})/\boldsymbol{\lambda}_{\text{Interpretation}}\\
\text{Attention} &= (\boldsymbol{\omega}_{\text{Emotion,Attention}}\text{Emotion} + \boldsymbol{\omega}_{\text{Relation,Attention}}\text{Relation}\\
&\quad + \boldsymbol{\omega}_{\text{Opinion,Attention}}\text{Opinion}\\
&\quad + \boldsymbol{\omega}_{\text{Interpretation,Attention}}\text{Interpretation})/\boldsymbol{\lambda}_{\text{Attention}}\\
\text{Arousal} &= (\boldsymbol{\omega}_{\text{Emotion,Arousal}}\text{Emotion} + \boldsymbol{\omega}_{\text{Relation,Arousal}}\text{Relation}\\
&\quad + \boldsymbol{\omega}_{\text{Opinion,Arousal}}\text{Opinion}\\
&\quad + \boldsymbol{\omega}_{\text{Interpretation,Arousal}}\text{Interpretation})/\boldsymbol{\lambda}_{\text{Arousal}}\\
\text{Sharing} &= (\boldsymbol{\omega}_{\text{Relation,Sharing}}\text{Relation} + \boldsymbol{\omega}_{\text{Interpretation,Sharing}}\text{Interpretation}\\
&\quad + \boldsymbol{\omega}_{\text{Arousal,Sharing}}\text{Arousal})/\boldsymbol{\lambda}_{\text{Sharing}}
\end{aligned}
$$

The linear equilibrium equations for the states other than Sharing can be solved in a symbolic manner to obtain explicit algebraic expressions for their equilibrium values (again the online WIMS Linear Solver tool was used); see Box 12.4. Here subscripts are abbreviated for the sake of briefness.

Box 12.4 Explicit algebraic solutions of the equilibrium equations of the second example network model; adopted from Schoenmaker et al. (2018)

$$
\begin{aligned}
&\textbf{Person} = X_1 = \mathrm{A}_1 \quad \textbf{Information} = X_2 = \mathrm{A}_2 \quad \textbf{Emotion} = X_3 = \mathrm{A}_3\\
&\textbf{Relation} = X_4 = \boldsymbol{\omega}_{\mathrm{P,R}}\mathrm{A}_1\\
&\textbf{Opinion} = X_5 = -[\mathrm{A}_1\boldsymbol{\omega}_{\mathrm{Int,O}}\boldsymbol{\omega}_{\mathrm{P,R}}(\boldsymbol{\lambda}_{\mathrm{Ar}}\boldsymbol{\lambda}_{\mathrm{At}}\boldsymbol{\omega}_{\mathrm{R,Int}}\\
&\qquad + \boldsymbol{\lambda}_{\mathrm{Ar}}\boldsymbol{\omega}_{\mathrm{At,Int}}\boldsymbol{\omega}_{\mathrm{RAt}} + \boldsymbol{\lambda}_{\mathrm{At}}\boldsymbol{\omega}_{\mathrm{Ar,Int}}\boldsymbol{\omega}_{\mathrm{R,Ar}})\\
&\qquad + \mathrm{A}_3(\boldsymbol{\lambda}_{\mathrm{Ar}}\boldsymbol{\omega}_{\mathrm{At,Int}}\boldsymbol{\omega}_{\mathrm{E,At}} + \boldsymbol{\lambda}_{\mathrm{At}}\boldsymbol{\omega}_{\mathrm{Ar,Int}}\boldsymbol{\omega}_{\mathrm{E,Ar}})\boldsymbol{\omega}_{\mathrm{Int,O}}\\
&\qquad + \mathrm{A}_2\boldsymbol{\omega}_{\mathrm{Inf,O}}((-\boldsymbol{\lambda}_{\mathrm{Ar}}\boldsymbol{\omega}_{\mathrm{At,Int}}\boldsymbol{\omega}_{\mathrm{Int,At}}) - \boldsymbol{\lambda}_{\mathrm{At}}\boldsymbol{\omega}_{\mathrm{Ar,Int}}\boldsymbol{\omega}_{\mathrm{Int,Ar}} + \boldsymbol{\lambda}_{\mathrm{Ar}}\boldsymbol{\lambda}_{\mathrm{At}}\boldsymbol{\lambda}_{\mathrm{I}})\\
&\qquad /[\boldsymbol{\omega}_{\mathrm{Int,O}}(\boldsymbol{\lambda}_{\mathrm{Ar}}\boldsymbol{\lambda}_{\mathrm{At}}\boldsymbol{\omega}_{\mathrm{O,Int}} + \boldsymbol{\lambda}_{\mathrm{Ar}}\boldsymbol{\omega}_{\mathrm{At,Int}}\boldsymbol{\omega}_{\mathrm{O,At}} + \boldsymbol{\lambda}_{\mathrm{At}}\boldsymbol{\omega}_{\mathrm{Ar,Int}}\boldsymbol{\omega}_{\mathrm{O,Ar}})\\
&\qquad + \boldsymbol{\lambda}_{\mathrm{O}}(\boldsymbol{\lambda}_{\mathrm{Ar}}\boldsymbol{\omega}_{\mathrm{At,Int}}\boldsymbol{\omega}_{\mathrm{Int,At}} + \boldsymbol{\lambda}_{\mathrm{At}}\boldsymbol{\omega}_{\mathrm{Ar,Int}}\boldsymbol{\omega}_{\mathrm{Int,Ar}} - \boldsymbol{\lambda}_{\mathrm{Ar}}\boldsymbol{\lambda}_{\mathrm{At}}\boldsymbol{\lambda}_{\mathrm{I}}))]
\end{aligned}
$$

$$\begin{aligned}
\textbf{Interpretation} = X_6 = -[&\mathrm{A}_1 \boldsymbol{\lambda}_{\mathrm{O}} \boldsymbol{\omega}_{\mathrm{P,R}} (\boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{R,Int}} + \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{At,Int}} \boldsymbol{\omega}_{\mathrm{R,At}} + \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{Ar,Int}} \boldsymbol{\omega}_{\mathrm{R,Ar}})\\
&+ \mathrm{A}_2 \boldsymbol{\omega}_{\mathrm{Inf,O}} (\boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{O,Int}} + \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{At,Int}} \boldsymbol{\omega}_{\mathrm{O,At}} + \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{Ar,Int}} \boldsymbol{\omega}_{\mathrm{O,Ar}})\\
&+ \mathrm{A}_3 \boldsymbol{\lambda}_{\mathrm{O}} (\boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{At,Int}} \boldsymbol{\omega}_{\mathrm{E,At}} + \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{Ar,Int}} \boldsymbol{\omega}_{\mathrm{E,Ar}})]\\
&/[\boldsymbol{\omega}_{\mathrm{Int,O}} (\boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{O,Int}} + \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{At,Int}} \boldsymbol{\omega}_{\mathrm{O,At}} + \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{Ar,Int}} \boldsymbol{\omega}_{\mathrm{O,Ar}})\\
&+ \boldsymbol{\lambda}_{\mathrm{O}} (\boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{At,Int}} \boldsymbol{\omega}_{\mathrm{Int,At}} + \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{Ar,Int}} \boldsymbol{\omega}_{\mathrm{Int,Ar}} - \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\lambda}_{\mathrm{I}})]
\end{aligned}$$

$$\begin{aligned}
\textbf{Attention} = X_7 = -[&\mathrm{A}_1 \boldsymbol{\omega}_{\mathrm{P,R}} (\boldsymbol{\omega}_{\mathrm{Int,O}} (\boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{O,At}} \boldsymbol{\omega}_{\mathrm{R,Int}}\\
&+ \boldsymbol{\omega}_{\mathrm{Ar,Int}} (\boldsymbol{\omega}_{\mathrm{O,At}} \boldsymbol{\omega}_{\mathrm{R,Ar}} - \boldsymbol{\omega}_{\mathrm{O,Ar}} \boldsymbol{\omega}_{\mathrm{R,At}}) - \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{O,Int}} \boldsymbol{\omega}_{\mathrm{R,At}})\\
&+ \boldsymbol{\lambda}_{\mathrm{O}} (\boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{Int,At}} \boldsymbol{\omega}_{\mathrm{R,Int}}\\
&+ \boldsymbol{\omega}_{\mathrm{Ar,Int}} (\boldsymbol{\omega}_{\mathrm{Int,At}} \boldsymbol{\omega}_{\mathrm{R,Ar}} - \boldsymbol{\omega}_{\mathrm{Int,Ar}} \boldsymbol{\omega}_{\mathrm{R,At}}) + \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\lambda}_{\mathrm{I}} \boldsymbol{\omega}_{\mathrm{R,At}}))\\
&+ \mathrm{A}_3 (\boldsymbol{\omega}_{\mathrm{Int,O}} (\boldsymbol{\omega}_{\mathrm{Ar,Int}} (\boldsymbol{\omega}_{\mathrm{E,Ar}} \boldsymbol{\omega}_{\mathrm{O,At}} - \boldsymbol{\omega}_{\mathrm{E,At}} \boldsymbol{\omega}_{\mathrm{O,Ar}}) - \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{E,At}} \boldsymbol{\omega}_{\mathrm{O,Int}})\\
&+ \boldsymbol{\lambda}_{\mathrm{O}} (\boldsymbol{\omega}_{\mathrm{Ar,Int}} (\boldsymbol{\omega}_{\mathrm{E,Ar}} \boldsymbol{\omega}_{\mathrm{Int,At}} - \boldsymbol{\omega}_{\mathrm{E,At}} \boldsymbol{\omega}_{\mathrm{Int,At}}) + \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\lambda}_{\mathrm{I}} \boldsymbol{\omega}_{\mathrm{E,At}}))\\
&+ \mathrm{A}_2 \boldsymbol{\omega}_{\mathrm{Inf,O}} (\boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{Int,At}} \boldsymbol{\omega}_{\mathrm{O,Int}}\\
&+ \boldsymbol{\omega}_{\mathrm{Ar,Int}} (\boldsymbol{\omega}_{\mathrm{Int,At}} \boldsymbol{\omega}_{\mathrm{O,Ar}} - \boldsymbol{\omega}_{\mathrm{Int,Ar}} \boldsymbol{\omega}_{\mathrm{O,At}}) + \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\lambda}_{\mathrm{I}} \boldsymbol{\omega}_{\mathrm{O,At}})]\\
&/[\boldsymbol{\omega}_{\mathrm{Int,O}} (\boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{O,Int}} + \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{At,Int}} \boldsymbol{\omega}_{\mathrm{O,At}} + \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{Ar,Int}} \boldsymbol{\omega}_{\mathrm{O,Ar}})\\
&+ \boldsymbol{\lambda}_{\mathrm{O}} (\boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{At,Int}} \boldsymbol{\omega}_{\mathrm{Int,At}} + \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{Ar,Int}} \boldsymbol{\omega}_{\mathrm{Int,Ar}} - \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\lambda}_{\mathrm{I}})]
\end{aligned}$$

$$\begin{aligned}
\textbf{Arousal} = X_8 = -[&\mathrm{A}_1 \boldsymbol{\omega}_{\mathrm{P,R}} (\boldsymbol{\omega}_{\mathrm{Int,O}} (\boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{O,Ar}} \boldsymbol{\omega}_{\mathrm{R,Int}}\\
&+ \boldsymbol{\omega}_{\mathrm{At,Int}} (\boldsymbol{\omega}_{\mathrm{O,Ar}} \boldsymbol{\omega}_{\mathrm{R,At}} - \boldsymbol{\omega}_{\mathrm{O,At}} \boldsymbol{\omega}_{\mathrm{R,Ar}}) - \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{O,Int}} \boldsymbol{\omega}_{\mathrm{R,Ar}})\\
&+ \boldsymbol{\lambda}_{\mathrm{O}} (\boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{Int,Ar}} \boldsymbol{\omega}_{\mathrm{R,Int}}\\
&+ \boldsymbol{\omega}_{\mathrm{At,Int}} (\boldsymbol{\omega}_{\mathrm{Int,Ar}} \boldsymbol{\omega}_{\mathrm{R,At}} - \boldsymbol{\omega}_{\mathrm{Int,At}} \boldsymbol{\omega}_{\mathrm{R,Ar}}) + \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\lambda}_{\mathrm{I}} \boldsymbol{\omega}_{\mathrm{R,Ar}}))\\
&+ \mathrm{A}_3 (\boldsymbol{\omega}_{\mathrm{Int,O}} (\boldsymbol{\omega}_{\mathrm{At,Int}} (\boldsymbol{\omega}_{\mathrm{E,At}} \boldsymbol{\omega}_{\mathrm{O,Ar}} - \boldsymbol{\omega}_{\mathrm{E,Ar}} \boldsymbol{\omega}_{\mathrm{O,At}}) - \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{E,Ar}} \boldsymbol{\omega}_{\mathrm{O,Int}})\\
&+ \boldsymbol{\lambda}_{\mathrm{O}} (\boldsymbol{\omega}_{\mathrm{At,Int}} (\boldsymbol{\omega}_{\mathrm{E,At}} \boldsymbol{\omega}_{\mathrm{Int,Ar}} - \boldsymbol{\omega}_{\mathrm{E,Ar}} \boldsymbol{\omega}_{\mathrm{Int,At}}) + \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\lambda}_{\mathrm{I}} \boldsymbol{\omega}_{\mathrm{E,Ar}}))\\
&+ \mathrm{A}_2 \boldsymbol{\omega}_{\mathrm{Inf,O}} (\boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{Int,Ar}} \boldsymbol{\omega}_{\mathrm{O,Int}}\\
&+ \boldsymbol{\omega}_{\mathrm{At,Int}} (\boldsymbol{\omega}_{\mathrm{Int,Ar}} \boldsymbol{\omega}_{\mathrm{O,At}} - \boldsymbol{\omega}_{\mathrm{Int,At}} \boldsymbol{\omega}_{\mathrm{O,Ar}}) + \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\lambda}_{\mathrm{I}} \boldsymbol{\omega}_{\mathrm{O,Ar}})]\\
&/[\boldsymbol{\omega}_{\mathrm{Int,O}} (\boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{O,Int}} + \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{At,Int}} \boldsymbol{\omega}_{\mathrm{O,At}} + \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{Ar,Int}} \boldsymbol{\omega}_{\mathrm{O,Ar}})\\
&+ \boldsymbol{\lambda}_{\mathrm{O}} (\boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\omega}_{\mathrm{At,Int}} \boldsymbol{\omega}_{\mathrm{Int,At}} + \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\omega}_{\mathrm{Ar,Int}} \boldsymbol{\omega}_{\mathrm{Int,Ar}} - \boldsymbol{\lambda}_{\mathrm{Ar}} \boldsymbol{\lambda}_{\mathrm{At}} \boldsymbol{\lambda})]
\end{aligned}$$

As can be seen, each of the equilibrium values is a linear combination of the three values A_1, A_2, A_3 (as predicted by Theorem 8), where the coefficients are expressed in terms of specific connection weights and scaling factors. For example, this means that if all of these values A_1, A_2, A_3 are reduced by 20%, all equilibrium values will be reduced by 20%. This indeed is the case in simulation examples. If the values of the connection weights and scaling factors are assigned as in the role matrices in Box 12.5, then the outcomes of the equilibrium values are (here the italic digits are repetitive):

$$\begin{aligned}
\text{Person} &= X_1 = \text{A}_1 \quad \text{Information} = X_2 = \text{A}_2 \quad \text{Emotion} = X_3 = \text{A}_3\\
\text{Relation} &= X_4 = \text{A}_1\\
\text{Opinion} &= X_5 = 0.17307692\text{A}_3 + 0.682692307\text{A}_2 + 0.1442307692\text{A}_1\\
\text{Interpretation} &= X_6 = 0.40384615\text{A}_3 + 0.259615384\text{A}_2 + 0.336538461\text{A}_1\\
\text{Attention} &= X_7 = 0.65384615\text{A}_3 + 0.13461538\text{A}_2 + 0.21153846\text{A}_1\\
\text{Arousal} &= X_8 = 0.65384615\text{A}_3 + 0.13461538\text{A}_2 + 0.21153846\text{A}_1
\end{aligned}$$

Box 12.5 Example values for the connection weights, adopted from Schoenmaker et al. (2018)

mb base connectivity	1	2	3	4
X_1				
X_2				
X_3				
X_4	X_1			
X_5	X_2	X_6		
X_6	X_4	X_5	X_7	X_8
X_7	X_3	X_4	X_5	X_6
X_8	X_3	X_4	X_5	X_6
X_9	X_4	X_6	X_8	

mcw connection weights	1	2	3	4
X_1				
X_2				
X_3				
X_4	1			
X_5	1	0.75		
X_6	0.5	0.75	0.75	0.75
X_7	1	0.25	0.25	0.25
X_8	1	0.25	0.25	0.25
X_9	0.5	1	1	

mcfw combination function weights	1 eucl	2 alogistic
X_1	1	
X_2	1	
X_3	1	
X_4	1	
X_5	1	
X_6	1	
X_7	1	
X_8	1	
X_9		1

mcfp function	1 eucl		2 alogistic	
parameter	1 n	2 λ	1 σ	2 τ
X_1	1	1		
X_2	1	1		
X_3	1	1		
X_4	1	1		
X_5	1	1.75		
X_6	1	2.75		
X_7	1	1.75		
X_8	1	1.75		
X_9			2.5	1.25

ms speed factors	1
X_1	0
X_2	0
X_3	0
X_4	0.5
X_5	0.5
X_6	0.5
X_7	0.5
X_8	0.5
X_9	0.5

iv initial values	1
X_1	A_1
X_2	A_2
X_3	A_3
X_4	0
X_5	0
X_6	0
X_7	0
X_8	0
X_9	0

It can be seen that each of these equilibrium state values is a weighted average of A_1, A_2, and A_3 (for each the sum of these weights is 1, as predicted by Theorem 8). Therefore, in particular, when all A_i are 1, all of these outcomes are 1. If only A_1 and A_2 are 1, then the outcomes depend just on the emotional charge A_3:

$$\begin{aligned}
\text{Person} = X_1 &= 1.0\,\text{Information} = X_2 = 1.0\,\text{Emotion} = X_3 = A_3\\
\text{Relation} = X_4 &= 1.0\\
\text{Opinion} = X_5 &= \ 0.17307692A_3 + \ 0.82692307\\
\text{Interpretation} = X_6 &= \ 0.40384615A_3 + \ 0.59615384\\
\text{Attention} = X_7 &= \ 0.6538461A_3 + \ 0.3461538\\
\text{Arousal} = X_8 &= \ 0.6538461A_3 + \ 0.3461538\\
\text{Sharing} = X_9 &= \mathbf{alogistic}_{\sigma,\tau}(0.5, 0.40384615A_3\\
&\quad + 0.59615384, 0.6538461A_3 + 0.3461538)
\end{aligned}$$

It can be seen from this analysis that the equilibrium values of Attention and Arousal depend for about 65% on the emotional charge level and as a consequence, the impact of the emotional charge on the equilibrium value of Interpretation is about 40%. The effect of emotional charge on Sharing works through two causal pathways: via Interpretation and via Arousal. This leads to the function

$$\begin{aligned}
\textbf{Sharing} = \mathbf{alogistic}_{\sigma,\tau}&(0.5, 0.40384615A_3\\
&+ 0.59615384, 0.6538461A_3 + 0.3461538)
\end{aligned}\tag{12.6}$$

of A_3, which is a monotonically increasing function of A_3.

12.9 Discussion

To analyse and predict from its structure what behaviour a given network model will eventually show is in general a challenging issue. For example, do all states in the network eventually converge to the same value? Some results are available for the case of acyclic, fully connected or strongly connected networks and for linear combination functions only; e.g., Bosse et al. (2015). It is often believed that when nonlinear functions are used, such results become impossible. Also, networks that are not strongly connected are often not addressed as they are more difficult to handle. This chapter shows what is still possible beyond the case of linear combination functions and also beyond the case of strongly connected networks. Parts of this chapter were adopted from Treur (2018b).

In this chapter general theorems were presented that relate network behaviour to the network structure characteristics. The relevant network structure characteristics concern two types of them:

- Network connectivity characteristics in terms of the network's strongly connected components and their mutual connections as shown in the network's condensation graph
- Network aggregation characteristics in terms of the combination functions used to aggregate the effects of multiple incoming connections (in particular, monotonicity, scalar-freeness, and normalisation).

The first item makes the approach applicable to any type of network connectivity, thus going beyond the limitation to strongly connected networks. The second item makes the approach applicable to a wider class of combination functions (most of which are nonlinear) going beyond the limitation to linear functions. However, there are also nonlinear functions that are not covered by this class. Some examples not covered are logistic functions, discrete threshold functions, and boolean functions, for example, as used in Karlsen and Moschoyiannis (2018), Watts (2002). The current chapter provides a first step to cover certain types of nonlinear functions. Nonlinear functions not covered yet form a next challenge that has been left open for now. In future research also other types of nonlinear functions will be explored further. Note that the notion of temporal-causal network itself is not a limitation as it is a very general notion which covers all types of discrete or smooth dynamical systems, and all systems of first-order differential equations. For these results, see Treur (2017), building further, among others, on Ashby (1960) and Port and van Gelder (1995).

The presented theorems subsume and generalise existing theorems for specific cases such as similar theorems for acyclic networks, fully connected networks and strongly connected networks (e.g., Theorems 3 and 4 in Sect. 12.6), and theorems addressing only linear combination functions as one fixed type of combination function; e.g., Theorem 3 at p. 120 of Bosse et al. (2015).

The theorems can be applied to predict behaviour of a given network, or to determine initial values in order to get some expected behaviour. In particular, they can be used as a method of verification to check the correctness of the implementation of a network. If simulation outcomes contradict the implications of the theorems, then some debugging of the implementation may be needed.

As already indicated in the Introduction section, after having developed the theorems presented here, it has turned out that these contributions also have some relations to research conducted from a different angle, namely on control of networks; e.g., Liu et al. (2011, 2012), Moschoyiannis et al. (2016), Haghighi and Namazi (2015), Karlsen and Moschoyiannis (2018). In that area, e.g., Liu et al. (2011, 2012), usually a system of linear differential equations is used for the dynamics of the considered network with N nodes $x_1, \ldots, x_N$ represented over time t by states $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))$. The dynamics is based on the connections with weights a_{ij} from x_j to x_i, overall represented by a matrix $\mathbf{A} = (a_{ij})$. For the control M additional nodes $u_1, \ldots, u_M$ are added, which are numerically represented over time t by states $\mathbf{u}(t) = (u_1(t), \ldots, u_M(t))$. These are meant to provide input at all time points in order to affect some of the network states (called drivers) over time. The latter nodes have connections to these driver nodes represented by an $N \times M$ input matrix $\mathbf{B} = (b_{ij})$ where b_{ij} represents the weight of the connection from node u_j to node x_i. Then overall the dynamics of the extended network can be represented as

$$\mathrm{d}\mathbf{x}(\mathrm{t})/\mathrm{dt} = \mathbf{A}\mathbf{x}(\mathrm{t}) + \mathbf{B}\mathbf{u}(\mathrm{t})$$

Reachability within the network relates to the powers of matrix $\mathbf{A}$ and controllability of the network from the states $u_1, \ldots, u_M$ relates to the combined $N \times NM$ matrix $\mathrm{C} = (\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{N-1}\mathbf{B}) \in \mathrm{R}^{N \times NM}$. Although precise mathematical criteria e.g., Kalman (1963) exist for this matrix $\mathbf{C}$ characterizing controllability of the network, such criteria often cannot be applied in practice as they depend on the precise values of the connection weights a_{ij} and in practical contexts usually these are not known. Therefore in literature such as (Liu et al. 2011, 2012) these weights are considered parameters, which introduces some complications: criteria for certain slightly different forms of (called structural) controllability are expressed in relation to these parameters e.g., Lin (1974); however, such criteria apply to (by far) most but not exactly all of such linear systems.

In contrast to the network control approach sketched in the previous paragraph, in the approach presented in the current chapter the lack of knowledge of specific weight values is not an issue, as these specific values are not used. Moreover, the theorems and their proofs do not make use of linearity assumptions, but instead of identified properties of a wider class of functions also including (a subset of the class of) nonlinear functions. Another difference is that the angle of controlling a network was not addressed in the current chapter, as the focus was on an angle of verification of a network model. However, some of the theorems still can be used for controlling a network. For example, Theorem 6 can be applied when the states $u_1, \ldots, u_M$ of the vector $\mathbf{u}$ in the above formalisation get outgoing connections (represented in matrix $\mathbf{B}$) to the states within the level 0 components in the original network. Then the states within the level 0 components in the original network are used as drivers. More specifically, this theorem provides the following results for the considered class of nonlinear functions extending the class of linear functions:

- Theorem 6(a) and (b) show that the whole network can be controlled by only controlling the final equilibrium values of the states within the level 0 components of the network. This actually can be done by extending the network by nodes $\boldsymbol{u}_i$ that are connected to the states in level 0 components of the original network. In the extended network this leads to singleton level 0 components $\{u_i\}$ and the other levels are increased by 1; for example, the level 0 components in the original network now become level 1 components in the extended network. Then from Theorem 6(a) and (b) it follows that the equilibrium values of all states in the network depend on the equilibrium values of the states u_i in the level 0 components $\{u_i\}$, and these equilibrium values are $\lim_{t\to\infty} u_i(t)$, $i = 1, \ldots, M$.
- Note that if the u_i are kept constant over time, these limit values of the u_i are just the initial values $u_i(0)$; in this case Theorem 6(c) and (d) apply. For example, for

this case Theorem 6(c) shows that if these initial values $u_i(0)$ are all set at 1, then after some time all states of the network will get equilibrium value 1.

This illustrates how all states of the network can be controlled by only controlling the states within the level 0 components. Note that this has a partial overlap with what is found in Liu et al. (2012) for the linear case, where also a decomposition based on the network's strongly connected components is used. In Theorems 7 to 9 above it is described that some more can be said about how exactly the equilibrium value of each of the network's nodes depends on the initial or final values of the states in the level 0 components. In particular for the linear case, this equilibrium value of each state of the network is a linear function of the initial or equilibrium values of the states in the level 0 components.

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